Topologies related to (I)-envelopes

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The (I)-envelope of a set A in a dual Banach space is defined by

$$\text{(I)-env}(A) = \bigcap \left\{ \overline{\bigcup_{n\in\mathbb{N}} \overline{\operatorname{conv} A_n}^{w^*}}^{\|\cdot\|} : A = \bigcup_{n\in\mathbb{N}} A_n \right\}.$$

This notion, inspired by the notion of (I)-generation from [2], was introduced in [3]. It was used in [4, 1, 5], in particular to characterize Grothendieck property and its quantitative version. (I)-env(A) is a norm-closed convex set and $\overline{\operatorname{conv} A}^{\|\cdot\|} \subset \text{(I)-env}(A) \subset \overline{\operatorname{conv} A}^{w^*}$ for any set A. We will address the following natural problem:

Question. Let X be a Banach space. Is there a (locally convex) topology τ on X^* such that (I)-env$(A) = \overline{\operatorname{conv} A}^{\tau}$ for each $A \subset X^*$?

The answer to the 'locally convex' version is 'sometimes yes, sometimes no', but a complete characterization is still missing. The 'topological' version is widely open and is connected to several interesting intermediate topologies on X^*.

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