Amenability, optimal transport and abstract ergodic theorems

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Using tools from the theory of optimal transport, four results concerning isometric actions of amenable topological groups with potentially unbounded orbits are established. Specifically, consider an amenable topological group G with no non-trivial homomorphisms to $\mathbb{R}$. If d is a compatible left-invariant metric on G, $E \subseteq G$ is a finite subset and $\epsilon > 0$, there is a finitely supported probability measure β on G so that

$$\max_{g,h \in E} \mathsf{W}(\beta g, \beta h) < \epsilon,$$

where W denotes the *Wasserstein* or *optimal transport* distance between probability measures on the metric space (G, d). When d is the word metric on a finitely generated group G, this strengthens a well known theorem of H. Reiter. Furthermore, when G is locally compact second countable, β may be replaced by an appropriate probability density $f \in L^1(G)$.

Also, when $G \curvearrowright X$ is a continuous isometric action on a metric space, the space of Lipschitz functions on the quotient $X /\!\!/ G$ is isometrically isomorphic to a 1-complemented subspace of the Lipschitz functions on X. And finally every continuous affine isometric action of G on a Banach space has a canonical invariant linear subspace. These results generalise previous theorems due to Schneider–Thom and Cúth–Doucha.

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