Planar embeddings of unimodal inverse limit spaces

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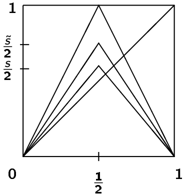
Unimodal map

Continuous map $f\colon [0,1] \to [0,1]$ is called *unimodal* if there exists a unique *critical point* c such that $f|_{[0,c)}$ is strictly increasing, $f|_{(c,1]}$ is strictly decreasing and $f(0) = f(1) = 0$.

Prototype - *tent map family* $\{T_s : s \in [0,2]\}$

$$T_s(x) := \begin{cases} sx, x \in [0,1/2] \\ s(1-x), x \in [1/2,1]. \end{cases}$$



For unimodal map T define *inverse limit space* as

$$X := \varprojlim([0,1], T) := \{(\dots, x_{-2}, x_{-1}, x_0) : x_i \in [0,1], T(x_{i-1}) = x_i\}$$

equipped with the topology of the Hilbert cube.

Planar embeddings of chainable continua

Continuum is a compact, connected metric space.

A *chain* is a finite collection of open sets $\mathcal{C} := \{\ell_i\}_{i=1}^n$ such that the *links* ℓ_i satisfy $\ell_i \cap \ell_j \neq \emptyset$ if and only if $|i - j| \leq 1$.

Chain is called ε*-chain* if the links are of diameter less than ε.

Continuum is *chainable* if it can be covered by an ε-chain for every $\varepsilon > 0$.

Theorem (R. H. Bing 1951.)

Every chainable continuum can be embedded in the plane.

Theorem (J. R. Isbell, 1959.)

Continuum is chainable iff it is inverse limit of a sequence of arcs.

Explicit construction of planar embeddings of UILs

- Brucks and Diamond (1995) - planar embeddings using symbolic description of UILs
- Bruin (1999) - embeddings are constructed such that the shift homeomorphism extends to a Lipschitz map on $\mathbb{R}^2$. (Barge, Martin, 1990., Boyland, de Carvalho, Hall 2012.)

Shift homeomorphism $\sigma\colon X \to X$, $\sigma((\dots, x_0)) := (\dots, x_0, T(x_0))$

Question(s) (Boyland 2015.)

Can a complicated X be embedded in $\mathbb{R}^2$ in multiple ways? YES!
Such that the shift-homeomorphism can be continuously extended to the plane? OPEN!

Equivalence of planar embeddings

Definition

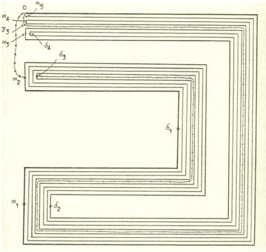
Denote two planar embeddings of X by $g_1 : X \to E_1 \subset \mathbb{R}^2$ and $g_2 : X \to E_2 \subset \mathbb{R}^2$. We say that g_1 and g_2 are *equivalent embeddings* if there exists a homeomorphism $h \colon E_1 \to E_2$ which can be extended to a homeomorphism of the plane.

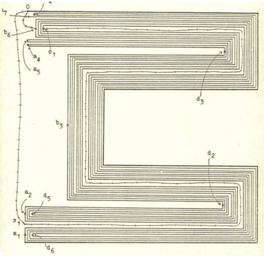
Definition

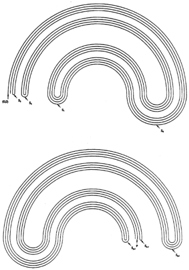
A point $a \in X \subset \mathbb{R}^2$ is *accessible (i.e., from the complement of* X*)* if there exists an arc $A = [x, y] \subset \mathbb{R}^2$ such that $a = x$ and $A \cap X = \{a\}$. We say that a composant $\mathcal{U} \subset X$ is *accessible*, if $\mathcal{U}$ contains an accessible point.

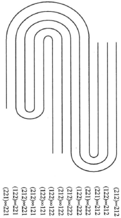
Knaster continuum K - full unimodal map

- Mayer (1983) - uncountably many non-equivalent planar embeddings of K with the same prime end structure and same set of accessible points.
- Mahavier (1989) - for every composant $\mathcal{U} \subset K$ there exists a planar embedding of K such that each point of $\mathcal{U}$ is accessible
- Schwartz (1992, PhD thesis) - uncountably many non-equivalent planar embeddings of K
- Dębski & Tymchatyn (1993) - study of accessibility in generalized Knaster continua









Results

For every (*not renormalizable, no wandering intervals*) unimodal map we obtain uncountably many embeddings by making an arbitrary point accessible.

Theorem (A., Bruin, Činč, 2016)

For every point $a \in X$ there exists an embedding of X in the plane such that a is accessible.

Every homeomorphism $h\colon X \to X$ is isotopic to σ^R for some $R \in \mathbb{Z}$ (Bruin & Štimac, 2012).

Corollary

There are uncountably many non-equivalent embeddings of X in the plane.

Symbolic description

Itinerary of a point $x \in [0,1]$ is $I(x) := \nu_0(x)\nu_1(x)\ldots$, where

$$\nu_i(x) := \begin{cases} 0, & T^i(x) \in [0,c], \\ 1, & T^i(x) \in [c,1]. \end{cases}$$

The kneading sequence is $\nu = I(T(c)) = c_1 c_2 c_3 \ldots$.
We say that a sequence $(s_i)_{i\geq 0}$ is *admissible* if it is realized as an itinerary of some point $x \in [0,1]$
Define $\Sigma_{adm} := \{(s_i)_{i\in\mathbb{Z}} : s_k s_{k+1} \ldots \text{admissible for every } k \in \mathbb{Z}\}$.
Then $X \simeq \Sigma_{adm}/\sim$, where $s \sim t \Leftrightarrow s_i = t_i$ for every $i \in \mathbb{Z}$, or if there exists $k \in \mathbb{Z}$ such that $s_i = t_i$ for all $i \neq k$ but $s_k \neq t_k$ and $s_{k+1}s_{k+2}\ldots = t_{k+1}t_{k+2}\ldots = \nu$.
Topology on the sequence space: $d((s_i)_{i\in\mathbb{Z}}, (t_i)_{i\in\mathbb{Z}}) := \sum_{i\in\mathbb{Z}} \frac{|s_i - t_i|}{2^{|i|}}$.

Basic arcs

Let $\overleftarrow{s} = \dots s_{-2}s_{-1}. \in \{0,1\}^{-\mathbb{N}}$ be an admissible left-infinite sequence (*i.e.*, every finite subword is admissible).
Basic arc (may be degenerate) is

$$A(\overleftarrow{s}) := \{x \in X : \nu_i(x) = s_i, \forall i < 0\} \subset X$$

$$\tau_L(\overleftarrow{s}) := \sup\{n > 1 : s_{-(n-1)} \dots s_{-1} = c_1 c_2 \dots c_{n-1}, \#_1(c_1 \dots c_{n-1}) \text{ odd}\}$$

$$\tau_R(\overleftarrow{s}) := \sup\{n \geq 1 : s_{-(n-1)} \dots s_{-1} = c_1 c_2 \dots c_{n-1}, \#_1(c_1 \dots c_{n-1}) \text{ even}\},$$

where $\#_1(a_1 \dots a_n)$ is a number of ones in a word $a_1 \dots a_n \subset \{0,1\}^n$

Lemma (Bruin, 1999.)

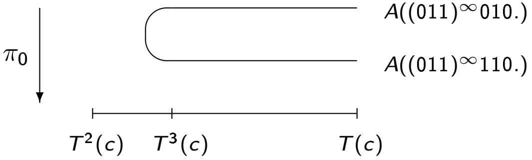
Let $\overleftarrow{s} \in \{0,1\}^{-\mathbb{N}}$ be admissible such that $\tau_L(\overleftarrow{s}), \tau_R(\overleftarrow{s}) < \infty$. Then

$$\pi_0(A(\overleftarrow{s})) = [T^{\tau_L(\overleftarrow{s})}(c), T^{\tau_R(\overleftarrow{s})}(c)].$$

If $\overleftarrow{t} \in \{0,1\}^{-\mathbb{N}}$ is another admissible left-infinite sequence such that $s_i = t_i$ for all $i < 0$ except for $i = -\tau_R(\overleftarrow{s}) = -\tau_R(\overleftarrow{t})$ (or $i = -\tau_L(\overleftarrow{s}) = -\tau_L(\overleftarrow{t})$), then $A(\overleftarrow{s})$ and $A(\overleftarrow{t})$ have a common boundary point.

Planar representation

Idea: draw every basic arc as horizontal arc in the plane, join the identified points by semi-circles. Horizontal arcs must be arranged such that semi-circles do not intersect and respecting the metric on symbol sequences!



Ordering on basic arcs

Definition (Ordering on basic arcs wrt L)

Let $L = \dots l_{-2} l_{-1}.$ be an admissible left-infinite sequence.
Let $\overleftarrow{s}, \overleftarrow{t} \in \{0,1\}^{-\mathbb{N}}$ and let $k \in \mathbb{N}$ be the smallest natural number such that $s_{-k} \neq t_{-k}$. Then

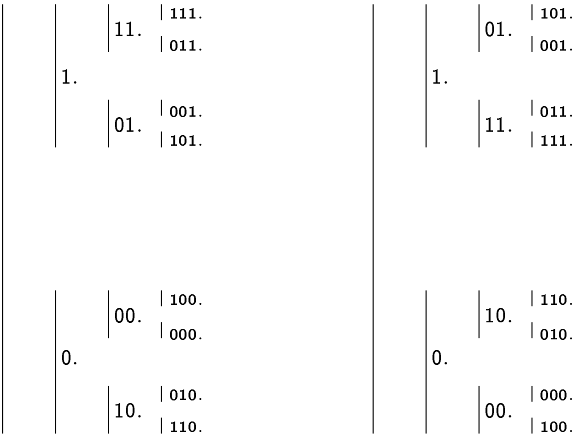
$$\overleftarrow{s} \prec_L \overleftarrow{t} \Leftrightarrow \begin{cases} t_{-k} = l_{-k} \text{ and } \#_1(s_{-(k-1)} \dots s_{-1}) - \#_1(l_{-(k-1)} \dots l_{-1}) \text{ even, or} \\ s_{-k} = l_{-k} \text{ and } \#_1(s_{-(k-1)} \dots s_{-1}) - \#_1(l_{-(k-1)} \dots l_{-1}) \text{ odd,} \end{cases}$$

where $\#_1(a_1 \dots a_n)$ is a number of ones in a word $a_1 \dots a_n \subset \{0,1\}^n$.

Let $\overleftarrow{s} \in \{0,1\}^{\mathbb{N}}$ be an admissible left-infinite sequence. Define $\psi_L : \{0,1\}^{-\mathbb{N}} \to C$ as

$$\psi_L(\overleftarrow{s}) := \sum_{i=1}^{\infty} (-1)^{\#_1(l_{-i} \dots l_{-1}) - \#_1(s_{-i} \dots s_{-1})} 3^{-i} + \frac{1}{2},$$

Note that $\psi_L(L) = 1$ is the largest point in C, where C is a middle-third Cantor set in $[0,1]$.



(*a*) (*b*)

Figure: (*a*) $L = \ldots 111.$ and (*b*) $L = \ldots 101.$

Embedding

Planar representation of a basic arc $A = A(\overleftarrow{s})$ is given as $(\pi_0(A), \psi_L(\overleftarrow{s}))$. Corresponding endpoints are joined by a semi-circle.

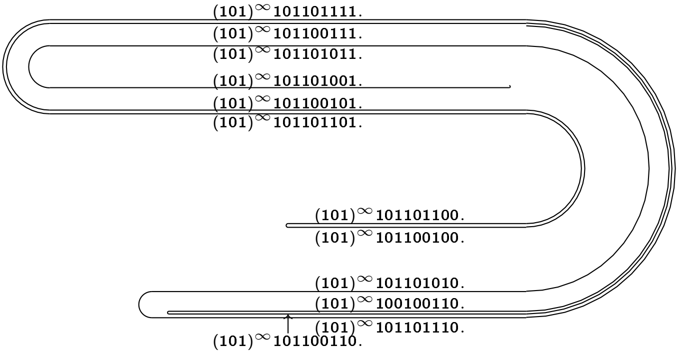


Figure: $\nu = 100110010\ldots$, $L = 1^\infty$.

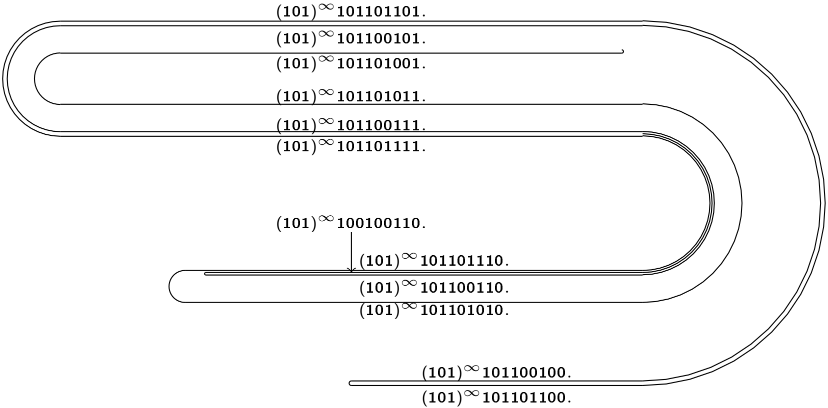
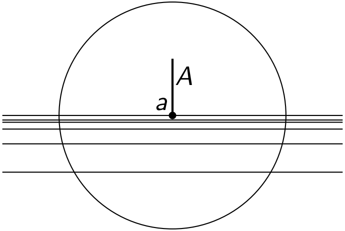


Figure: Embedding of the same arc as in the previous picture, with $L = (101)^\infty$.

Proof of the main theorem

Assume that $a = (\dots, a_{-1}, a_0) \in X$ is contained in a basic arc $A = A(\dots l_{-2} l_{-1})$. Consider the planar representation of X obtained by the ordering making $L = \dots l_{-2} l_{-1}.$ the largest. The point a is represented as $(a_0, 1)$.



Some accessibility results

An arc-component is called *fully-accessible* if every point in it is accessible.

- arc-component $\mathcal{U} \ni (\dots, 0, 0)$ is always fully-accessible (except in non-standard embeddings of Knaster continuum)
- for every unimodal inverse limit space we have constructed an embedding with exactly 1, 2, and 3 fully-accessible (non-degenerate) arc-components.
- for every $n \in \mathbb{N}$ there exists a chainable indecomposable planar continuum with exactly *n fully-accessible* composants (namely cores of $\nu = (10^{n-2}1)^{\infty}$ in Brucks-Diamond embedding).

Further research

- In some (recurrent) UILs there exist degenerate arc-components (Barge, Brucks, Diamond, 1996.). What happens if such point is embedded the largest? (We still cannot obtain symbolic representation of such points)
- (Nadler and Quinn 1972.) If X is chainable continuum and $x \in X$ is a point, does there exist a planar embedding of X such that x is accessible?
- (Mayer 1982.) Are there uncountably many inequivalent embeddings of every chainable indecomposable continuum (with the same set of accessible points and the same prime end structure?)
- prime ends, Wada channels ...

Thank you!