On (co)homology properties of remainders of Stone-Čech compactifications of metrizable spaces

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Abstract

In the paper the Čech border homology and cohomology groups of closed pairs of normal spaces are constructed and investigated. These groups give an intrinsic characterizations of Čech homology and cohomology groups based on finite open coverings, homological and cohomological coefficients of cyclicity and cohomological dimensions of remainders of Stone-Čech compactifications of metrizable spaces.

Keywords and Phrases: Čech homology, Čech cohomology, Stone-Čech compactification, remainder, cohomological dimension, coefficient of cyclicity.

Introduction

The motivation of the paper is the following problem:

Find necessary and sufficient conditions under which a space of given class has a compactification whose remainder has the given topological property (cf. [Sm_2], Problem I, p.332 and Problem II, p.334).

Many mathematicians investigated this problem:

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- J.M.Aarts [A], J.M.Aarts and T.Nishiura [A-N], Y. Akaike, N. Chinen and K. Tomoyasu [Ak-Chin-T], V.Baladze [B_1], M.G. Charalambous [Ch], A.Chigogidze ([Chi_1], [Chi_2]), H. Freudenthal ([F_1],[F_2]), K.Morita [Mo], E.G. Skljarenko [Sk], Ju.M.Smirnov ([Sm_1]-[Sm_5]) and H.De Vries [V] found conditions under which the spaces have extensions whose remainders have given covering and inductive dimensions and combinatorial properties.

- The remainders of finite order extensions is defined and investigated by H.Inasaridze ([I_1], [I_2]). Using in these papers obtained results author [I_3], L.Zambakhidze ([Z_1],[Z_2]) and I.Tsereteli [Ts] solved interesting problems of homological algebra, general topology and dimension theory.

- n-dimensional (co)homology groups and cohomotopy groups of remainders are studied by V.Baladze [B_3], V.Baladze and L.Turmanidze [B-Tu] and A.Calder [C].

- The characterizations of shapes of remainders of spaces established in papers of V.Baladze ([B_2],[B_3]), B.J.Ball [Ba], J.Keesling ([K_1], [K_2]), J.Keesling and R.B. Sher [K-Sh].

The paper is devoted to study this problem for the properties: Čech (co)homology groups based on finite open coverings, coefficient of cyclicities and cohomological dimensions of remainders of Stone-Čech compactifications of metrizable spaces are given groups and given numbers, respectively.

In this paper are defined the Čech type covariant and contravariant functors which coefficients in an abelian group G

$$\check{\mathrm{H}}_n^{\infty}(-,-;G) : \mathscr{N}^{\mathbf{2}} \to \mathscr{A}b$$

and

$$\hat{\mathrm{H}}^n_{\infty}(-,-;G) : \mathscr{N}^{\mathbf{2}} \to \mathscr{A}b$$

from the category $\mathscr{N}^2$ of closed pairs of normal spaces and proper maps to the category $\mathscr{A}b$ of abelian groups and homomorphisms. The construction of these functors are based on all border open coverings of pair $(X, A) \in ob(\mathscr{N}^{\mathbf{2}})$ (see Definition 1.1).

One of main results of paper is following (see Theorem 2.1). Let $\mathscr{M}^2$ be the category of closed pairs of metrizable spaces. For each closed pair

$(X, A) \in ob(\mathscr{M}^{\mathbf{2}})$

$$\check{H}_n^f(\beta X \setminus X, \beta A \setminus A; G) = \hat{H}_n^{\infty}(X, A; G)$$

and

$$\hat{H}_f^n(\beta X \setminus X, \beta A \setminus A; G) = \check{H}_{\infty}^n(X, A; G),$$

where $\check{H}_n^f(\beta X \setminus X, \beta A \setminus A; G)$ and $\hat{H}_f^n(\beta X \setminus X, \beta A \setminus A; G)$ are Čech homology and cohomology groups based on all finite open coverings of $(\beta X \setminus X, \beta A \setminus A)$, respectively (see [E-St], Ch. IX, p.237).

In the paper also are defined the border cohomological and homological coefficients of cyclicity η_G^{∞} and η_{∞}^G, border cohomological dimension $d_f^{\infty}(X; G)$ and proved the following relations (see Theorem 2.3 and Theorem 2.5):

$$\eta_G^{\infty}(X, A) = \eta_G(\beta X \setminus X, \beta A \setminus A),$$

$$\eta_{\infty}^G(X, A) = \eta^G(\beta X \setminus X, \beta A \setminus A),$$

$$d_f^{\infty}(X; G) \leq d_f(\beta X \setminus X; G),$$

where $\eta_G(\beta X \setminus X, \beta A \setminus A)$, $\eta^G(\beta X \setminus X, \beta A \setminus A)$ and $d_f(\beta X \setminus X; G)$ are cohomological coefficient of cyclicity [No], homological coefficient of cyclicity (see Definition 2.2) and small cohomological dimension [N] of remainders $(\beta X \setminus X, \beta A \setminus A)$ and $\beta X \setminus X$, respectively.

Without any specification we will use definitions, notions and results from books General Topology [En] and Algebraic Topology [E-St].

1 On Čech border homology and cohomology groups

In this section we give an outline of a generalization of Čech homology theory by replacing the set of all finite open coverings in the definition of Čech (co)homology group $(\hat{H}_f^n(X, A; G))$ $\check{H}_n^f(X, A; G)$ (see [E-St],Ch.IX, p.237) by a set of all finite open families with compact enclosures. For this aim here we give the following definition.

Definition 1.1. (Yu.M.Smirnov, [Sm$_4$]). A family $\alpha = \{U_1, U_2, \cdots, U_n\}$ of open sets of normal space X is called a border covering of X if its enclosure $K_\alpha = X \setminus \bigcup_{i=1}^{n} U_i$ is a compact subset of X.

An indexed family of sets in X is a function α from a indexed set V_α to the set 2^X of subsets of X. The image $\alpha(v)$ of index $v \in V_\alpha$ denote by α_v. Thus the indexed family α is the family $\alpha = \{\alpha_v\}_{v \in V_\alpha}$. If $|V_\alpha| < \aleph_0$, then we say that α family is a finite family.

Let A be a subset of X and V_α^A subset of V_α. A family $\{\alpha_v\}_{v \in V_\alpha^A}$ is called the subfamily of family $\{\alpha_v\}_{v \in V_\alpha}$.

The family $\alpha = \{\alpha_v\}_{v \in (V_\alpha, V_\alpha^A)}$ is called family of pair (X, A).

Definition 1.2. (cf.[Sm$_4$]). A finite open family $\alpha = \{\alpha_v\}_{v \in (V_\alpha, V_\alpha^A)}$ of pair (X, A) from the category $\mathscr{N}^{\mathbf{2}}$ is called a border covering of (X, A) if there exists a compact subset K_α of X such that $X \setminus K_\alpha = \bigcup\limits_{v \in V_\alpha} \alpha_v$ and $A \setminus K_\alpha \subseteq \bigcup\limits_{v \in V_\alpha^A} \alpha_v$.

The set of all border covers of (X, A) is denoted by $\mathrm{cov}_\infty(X, A)$. Let $K_\alpha^A = K_\alpha \cap A$. Then the family $\{\alpha_v \cap A\}_{v \in V_\alpha^A}$ is a border cover of subspace A.

Definition 1.3. Let $\alpha, \beta \in \mathrm{cov}_\infty(X, A)$ be two border coverings of (X, A) with indexing pairs (V_α, V_α^A) and (V_β, V_β^A), respectively. We say that the border covering β is a refinement of border covering α if there exists a refinement projection function $p : (V_\beta, V_\beta^A) \to (V_\alpha, V_\alpha^A)$ such that for each index $v \in V_\beta$ $(v \in V_\beta^A)$ $\beta_v \subset \alpha_{p(v)}$.

It is clear that $\mathrm{cov}_\infty(X, A)$ becomes a directed set with the relation $\alpha \leq \beta$ whenever β is a refinement of α.

Note that for each $\alpha \in \mathrm{cov}_\infty(X, A)$ $\alpha \leq \alpha$ and if for each $\alpha, \beta, \gamma \in \mathrm{cov}_\infty(X, A)$, $\alpha \leq \beta$ and $\beta \leq \gamma$, then $\alpha \leq \gamma$.

Let $\alpha, \beta \in \mathrm{cov}_\infty(X, A)$ be two border coverings with indexing pairs (V_α, V_α^A) and (V_β, V_β^A), respectively. Consider a family $\gamma = \{\gamma_v\}_{v \in (V_\gamma, V_\gamma^A)}$, where $V_\gamma = V_\alpha \times V_\beta$ and $V_\gamma^A = V_\alpha^A \times V_\beta^A$. Let $v = (v_1, v_2)$, where $v_1 \in V_\alpha$, $v_2 \in V_\beta$. Assume that $\gamma_v = \alpha_{v_1} \cap \beta_{v_2}$. The family $\gamma = \{\gamma_v\}_{v \in (V_\gamma, V_\gamma^A)}$ is a border covering of (X, A) and $\gamma \geq \alpha, \beta$.

For each border covering $\alpha \in \mathrm{cov}^\infty(\mathrm{X}, \mathrm{A})$ with indexing pair (V_α, V_α^A) by (X_α, A_α) denote the nerve α, where A_α is the subcomplex of simplexes s of complex X_α with vertices of V_α^A such that $\mathrm{Car}_\alpha(\mathrm{s}) \cap \mathrm{A} \neq \emptyset$. The pair (X_α, A_α) forms a simplicial pair. Besides, any two refinement projection functions $p, p^{'} : \beta \to \alpha$ induces contiguous simplicial maps of simplicial pairs $p_\alpha^\beta, q_\alpha^\beta : (X_\beta, A_\beta) \to (X_\alpha, A_\alpha)$ (see [E-St], pp.234-235).

Using the construction of formal homology theory of simplicial complexes ([E-St], Ch.VI) we can define the unique homomorphisms

$$p^{\beta}_{\alpha *} : H_n(X_\beta, A_\beta : G) \to H_n(X_\alpha, A_\alpha; G)$$

and

$$(p^{\beta *}_{\alpha} : H^n(X_\alpha, A_\alpha : G) \to H_n(X_\beta, A_\beta; G)),$$

where G is any abelian coefficient group.

Note that $p^{\alpha}_{\alpha *} = 1_{H_n(X_\alpha, A_\alpha : G)}$ and $p^{\alpha *}_{\alpha} = 1_{H^n(X_\alpha, A_\alpha : G)}$. If $\gamma \geq \beta \geq \alpha$ than

$$p^{\gamma}_{\alpha *} = p^{\beta}_{\alpha *} \cdot p^{\gamma}_{\beta *}$$

and

$$p^{\gamma *}_{\alpha} = p^{\gamma *}_{\beta} \cdot p^{\beta *}_{\alpha}.$$

Thus, the families

$$\{H_n(X_\alpha, A_\alpha; G), p^{\beta}_{\alpha *}, \mathrm{cov}_\infty(X, A)\}$$

and

$$\{H^n(X_\alpha, A_\alpha; G), p^{\beta *}_{\alpha}, \mathrm{cov}_\infty(X, A)\}$$

form the inverse and direct systems of groups.

The inverse and direct limit groups of above defined inverse and direct systems denote by symbols

$$\check{H}^{\infty}_{n}(X, A; G) = \varinjlim \{H_n(X_\alpha, A_\alpha; G), p^{\beta}_{\alpha *}, \mathrm{cov}_\infty(X, A)\}$$

and

$$\hat{H}^{n}_{\infty}(X, A; G) = \varprojlim \{H^n(X_\alpha, A_\alpha; G), p^{\beta *}_{\alpha}, \mathrm{cov}_\infty(X, A)\}$$

and call n-dimensional Čech border homology group and n-dimensional Čech border cohomology group of pair (X, A) with coefficients in abelian group G, respectively.

According to [E-St] a border covering $\alpha \in \mathrm{cov}_\infty(X, A)$ indexed by (V_α, V^A_α) is called proper if V^A_α is the set of all $v \in V_\alpha$ with $\alpha_v \cap A \neq \emptyset$. The set of proper border covering denote by $\mathrm{Pcov}_\infty(X, A)$. Now define a function

$$\rho : \mathrm{cov}_\infty(X) \to \mathrm{cov}_\infty(X, A)$$

By definition, for each border covering of X $\alpha = \{\alpha_v\}_{v \in V_\alpha}$

$$\rho(\alpha) = \{\alpha_v\}_{v \in (V_\alpha, V')},$$

where V' is the set of $v \in V_\alpha$ for which $\alpha_v \cap A \neq \emptyset$. It is clear that the family $\rho(\alpha)$ is a proper border cover and the function $\rho : \mathrm{cov}_\infty(X) \to \mathrm{Pcov}_\infty(X, A)$ induced by ρ is one to one. Besides, if $\alpha' \leq \alpha$, then $\rho(\alpha') \leq \rho(\alpha)$.

Proposition 1.4. For each pair $(X, A) \in ob(\mathscr{N}^2)$ the set $\mathrm{Pcov}_\infty(X, A)$ of proper border coverings of (X, A) is a cofinal subset of $\mathrm{cov}_\infty(X, A)$.

Proof. Let $\alpha = \{a_v\}_{v \in (V_\alpha, V_\alpha^A)}$ be a border covering of (X, A). Assume that

$$V' = \{\alpha_v | \alpha_v \cap A \neq \emptyset, v \in V_\alpha^A)\}.$$

Consider a family $\beta = \{\beta_v\}_{v \in (V_\alpha, V')}$ consisting of subsets

$$\beta_v = \alpha_v \setminus A, \qquad v \in V_\alpha \setminus V'$$

and

$$\beta_v = \alpha_v, \qquad v \in V'.$$

Note that β is a border covering of (X, A) and $\beta \geq \alpha$. □

Consequently, in definitions of Čech border homology and cohomology groups of pairs $(X, A) \in ob(\mathscr{N}^2)$ we may replace the set $\mathrm{cov}_\infty(X, A)$ by the subset $\mathrm{Pcov}_\infty(X, A)$.

Now we define for a given proper map $f : (X, A) \to (Y, B)$ of pairs the induced homomorphisms

$$f_* : \check{H}_n^\infty(X, A; G) \to \check{H}_n^\infty(Y, B; G)$$

and

$$f^* : \hat{H}^n_\infty(X, A; G) \to \hat{H}^n_\infty(Y, B; G).$$

Let $\alpha \in \mathrm{cov}_\infty(Y, B)$ be a border covering with index set V_α and $K_\alpha = Y \setminus \bigcup\limits_{v \in V_\alpha} \alpha_v$. Consider a family $\alpha' = \{f^{-1}(\alpha_v)\}_{v \in V_\alpha}$. Note that

$$X \setminus \bigcup_{v \in V_\alpha} f^{-1}(\alpha_v) = X \setminus f^{-1}(\bigcup_{v \in V_\alpha} \alpha_v) = X \setminus f^{-1}(Y \setminus K_\alpha) = f^{-1}(K_\alpha).$$

Let $\alpha_v^{'} = f^{-1}(\alpha_v)$ and $V_{\alpha'} = V_\alpha$. By condition $f^{-1}(K_\alpha)$ is a compact subset of X.

Since $B \setminus K_\alpha \subseteq \bigcup\limits_{v \in V_\alpha^B} \alpha_v$, the subfamily $\{f^{-1}(\alpha_v) | v \in V_\alpha^B\}$ is such that $A \setminus f^{-1}(K_\alpha) \subseteq \bigcup\limits_{v \in V_\alpha^B} f^{-1}(\alpha_v)$. Let $V_{\alpha'}^A = V_\alpha^B$ and $K_{\alpha'} = f^{-1}(K_\alpha)$. Note that $A \setminus K_{\alpha'} \subset \bigcup\limits_{v \in V_{\alpha'}^A} f^{-1}(\alpha_v)$. Hence, $\alpha^{'} = \{f^{-1}(\alpha_v)\}_{v \in (V_{\alpha'}, V_{\alpha'}^A)}$ is a border cover of pair (X, A).

It is clear that $X_{\alpha'}$ is a subcomplex of Y_α and $A_{\alpha'}$ is a subcomplex of B_α. By a symbol $f_\alpha : (X_{\alpha'}, A_{\alpha'}) \to (Y_\alpha, B_\alpha)$ denote the simplicial inclusion of $(X_{\alpha'}, A_{\alpha'})$ into (Y_α, B_α).

If $\alpha, \beta \in \mathrm{cov}_\infty(Y, B)$ and $\beta \geq \alpha$, then there exist the commutative diagrams

$$\begin{tikzcd}
H_n(X_{\beta'}, A_{\beta'}; G) \arrow[r, "f_{\beta *}"] \arrow[d, "p_{\alpha' *}^{\beta'}"'] & H_n(X_\beta, A_\beta; G) \arrow[d, "p_{\alpha *}^{\beta}"] \\
H_n(X_{\alpha'}, A_{\alpha'}; G) \arrow[r, "f_{\alpha *}"'] & H_n(X_\alpha, A_\alpha; G)
\end{tikzcd}$$

and

$$\begin{tikzcd}
H^n(X_\alpha, A_\alpha; G) \arrow[r, "f_\alpha^*"] \arrow[d, "p_\alpha^{\beta *}"'] & H^n(X_{\alpha'}, A_{\alpha'}; G) \arrow[d, "p_{\alpha'}^{\beta' *}"] \\
H^n(X_\beta, A_\beta; G) \arrow[r, "f_\beta^*"'] & H^n(X_{\beta'}, A_{\beta'}; G).
\end{tikzcd}$$

Thus, for each $\alpha \in \mathrm{cov}_\infty(Y, B)$ the induced homomorphisms $f_{\alpha *}$ and f_α^* together with function $\varphi : \mathrm{cov}_\infty(Y, B) \to \mathrm{cov}_\infty(X, A)$ given by formula

$$\varphi(\alpha) = f^{-1}(\alpha), \alpha \in \mathrm{cov}_\infty(Y, B)$$

form maps

$$(f_{\alpha *}, \varphi) : \{H_n(X_{\alpha'}, A_{\alpha'}), p_{\alpha' *}^{\beta'}, \mathrm{cov}_\infty(X, A)\} \to \{H_n(Y_\alpha, A_\alpha), p_{\alpha *}^{\beta}, \mathrm{cov}_\infty(Y, B)\}$$

and

$$(f_\alpha^*, \varphi) : \{H^n(Y_\alpha, A_\alpha), p_\alpha^{\beta *}, \mathrm{cov}_\infty(Y, B)\} \to \{H^n(X_{\alpha'}, A_{\alpha'}), p_{\alpha'}^{\beta'_*}, \mathrm{cov}_\infty(X, A)\}.$$

The limits of maps $(f_{\alpha *}, \varphi)$ and (f_α^*, φ) denote by

$$f_* : \check{H}_n^\infty(X, A; G) \to \check{H}_n^\infty(Y, B; G)$$

and

$$f^* : \hat{H}^n_\infty(Y, B; G) \to \hat{H}^n_\infty(X, A; G)$$

and call homomorphisms induced by proper map $f : (X, A) \to (Y, B)$.

Note that if $f : (X, A) \to (Y, B)$ is the identity map, then the induced homomorphisms $f_* : \check{H}_n^\infty(X, A; G) \to \check{H}_n^\infty(Y, B; G)$ and $f^* : \hat{H}^n_\infty(Y, B; G) \to \hat{H}^n_\infty(X, A; G)$ are the identity homomorphism. Besides, for each proper maps $f : (X, A) \to (Y, B)$ and $g : (Y, B) \to (Z, C)$

$$(g \cdot f)_* = g_* \cdot f_*$$

and

$$(g \cdot f)^* = f^* \cdot g^*.$$

We have the following theorem.

Theorem 1.5. *There exist the covariant and contravariant functors*

$$\check{\mathrm{H}}_*^\infty(-, -; G) : \mathscr{N}^2 \to \mathscr{A}b$$

and

$$\hat{\mathrm{H}}^*_\infty(-, -; G) : \mathscr{N}^2 \to \mathscr{A}b$$

given by formulas

$$\check{\mathrm{H}}_*^\infty(-, -; G)(X, A) = \check{H}_*^\infty(X, A; G), \quad (X, A) \in ob(\mathscr{N}^2)$$

$$\check{\mathrm{H}}_*^\infty(-, -; G)(f) = f_*, \quad f \in \mathrm{Mor}_{\mathscr{N}^2}((\mathrm{X}, \mathrm{A}), (\mathrm{Y}, \mathrm{B}))$$

and

$$\hat{\mathrm{H}}^*_\infty(-, -; G)(X, A) = \hat{H}^*_\infty(X, A; G), \quad (X, A) \in ob(\mathscr{N}^2)$$

$$\hat{\mathrm{H}}^*_\infty(-, -; G)(f) = f^*, \quad f \in \mathrm{Mor}_{\mathscr{N}^2}((\mathrm{X}, \mathrm{A}), (\mathrm{Y}, \mathrm{B})).$$

Proof. The proof follows from above given discussion. □

The functors $\check{\mathrm{H}}_*^\infty(-,-;G)$ and $\hat{\mathrm{H}}^*_\infty(-,-;G)$ we will call Čech border homology and cohomology functors, respectively.

Now we define boundary and coboundary homomorphisms

$$\partial_n : \check{H}_n^\infty(X,A;G) \to \check{H}_{n-1}^\infty(A;G)$$

and

$$\delta^n : \hat{H}_\infty^{n-1}(A;G) \to \hat{H}^n_\infty(X,A;G).$$

Let $(X,A) \in ob(\mathscr{N})$, $\beta, \alpha \in \mathrm{cov}_\infty(X,A)$ and $\beta \geq \alpha$. The refinement projection functions induce the unique homomorphisms $p^\beta_{\alpha *} : H_n(X,A;G) \to H_n(A;G)$ and $p^{\beta *}_\alpha : H^n(A_\alpha;G) \to H^n(A_\beta;G)$, which form the direct and inverse systems

$$\{H_n(A_\alpha;G), p^\beta_{\alpha *}, \mathrm{cov}_\infty(X,A)\}$$

and

$$\{H^n(A_\alpha;G), p^{\beta *}_\alpha, \mathrm{cov}_\infty(X,A)\}.$$

Let

$$\check{H}_n^\infty(A;G)_{(X,A)} = \varprojlim \{H_n(A_\alpha;G), p^\beta_{\alpha *}, \mathrm{cov}_\infty(X,A)\}$$

and

$$\hat{H}^n_\infty(A;G)^{(X,A)} = \varinjlim \{H^n(A_\alpha;G), p^{\beta *}_\alpha, \mathrm{cov}_\infty(X,A)\}.$$

Our main aim is to show that the groups $\check{H}_n^\infty(A;G)$ and $\hat{H}_n^\infty(A;G)_{(X,A)}$, $\hat{H}^n_\infty(A;G)$ and $\hat{H}^n_\infty(A;G)^{(X,A)}$ are isomorphical groups.

Now define a function $\varphi : \mathrm{cov}_\infty(X,A) \to \mathrm{cov}_\infty(A,\emptyset)$. Let $\alpha = \{\alpha_v\}_{v\in(V_\alpha,V_\alpha^A)} \in \mathrm{cov}_\infty(X,A)$. Assume that $(\varphi(\alpha))_v = \alpha_v \cap A$ for $v \in V_\alpha^A$. We have define the border covering $\varphi(\alpha) \in \mathrm{cov}_\infty(A,\emptyset)$ indexed by pair $(V_\alpha,\emptyset)$.

Let $K_\alpha = X \setminus \bigcup\limits_{v\in V_\alpha} \alpha_v$. Note that

$$A \setminus (K_\alpha \cap A) = \bigcup_{v \in V_\alpha^A} (\alpha_v \cap A) = \bigcup_{v\in V_\alpha^A} (\varphi(\alpha))_v.$$

It is clear that $K_\alpha \cap A$ is a compact subset of subspace A. Thus, $\varphi(\alpha) \in \mathrm{cov}_\infty(A,\emptyset)$. The defined function is the order preserving function.

It is easy to show that the image of function φ is a cofinal subset of set $\mathrm{cov}_\infty(A,\emptyset)$. Note that $A_\alpha = A_{\varphi(\alpha)}$. By $\varphi_\alpha : A_{\varphi(\alpha)} \to A_\alpha$ denote this simplicial isomorphism. Hence, the family of pairs $(\varphi_\alpha, \varphi)$ induces a map of inverse systems and direct systems

$$(\varphi_{\alpha*}, \varphi) : \{H_n(A_\alpha; G), p^\beta_{\alpha*}, \mathrm{cov}_\infty(A, \emptyset) \to \{H_n(A_\alpha; G), p^\beta_{\alpha*}, \mathrm{cov}_\infty(X, A)\}$$

and

$$(\varphi^*_\alpha, \varphi) : \{H^n(A_\alpha; G), p^{\beta*}_\alpha, \mathrm{cov}_\infty(X, A) \to \{H_n(A_\alpha; G), p^{\beta*}_\alpha, \mathrm{cov}_\infty(A, \emptyset)\}.$$

Let $\Phi^\infty_n = \varprojlim (\varphi_{\alpha*}, \varphi)$ and $\Phi^n_\infty = \varinjlim (\varphi^*_\alpha, \varphi)$. Since all homomorphisms $\varphi_{\alpha*}$ and φ^*_α are isomorphisms the defined limit homomorphisms

$$\Phi^\infty_n : \check{H}^\infty_n(A; G) \to \check{H}^\infty_n(A; G)_{(X,A)}$$

and

$$\Phi^n_\infty : \hat{H}^n_\infty(A; G)^{(X,A)} \to H^n_\infty(A; G)$$

are isomorphisms.

Now define a function $\psi : \mathrm{cov}_\infty(X, A) \to \mathrm{cov}_\infty(X, \emptyset)$. For each $\alpha = \{\alpha_v\}_{v \in (V_\alpha, V^A_\alpha)} \in \mathrm{cov}_\infty(X, A)$ assume that $(\psi(\alpha))_v = \alpha_v$, $v \in V_\alpha$. The family $\psi(\alpha)$ is indexed by $(V_\alpha, \emptyset)$ and $\psi(\alpha) \in \mathrm{cov}_\infty(X, \emptyset)$.

Note that $X_\alpha = X_{\psi(\alpha)}$. Let $\psi_\alpha : X_{\psi(\alpha)} \to X_\alpha$ be a simplicial isomorphism. The family of pairs (ψ_α, ψ) induce the maps of inverse and direct systems

$$(\psi_{\alpha*}, \psi) : \{H_n(X_\alpha; G), p^\beta_{\alpha*}, \mathrm{cov}_\infty(X, \emptyset)\} \to \{H_n(X_\alpha; G), p^\beta_{\alpha*}, \mathrm{cov}_\infty(X, A)\}$$

and

$$(\psi^*_\alpha, \psi) : \{H^n(X_\alpha; G), p^{\beta*}_\alpha, \mathrm{cov}_\infty(X, A)\} \to \{H^n(X_\alpha; G), p^{\beta*}_\alpha, \mathrm{cov}_\infty(X, \emptyset)\}.$$

Let $\Psi^\infty_n = \varprojlim (\psi_{\alpha*}, \psi)$ and $\Psi^n_\infty = \varinjlim (\psi^*_\alpha, \psi)$. Since each $\psi_{\alpha*}$ and ψ^*_α are isomorphisms the induced limit homomorphisms

$$\Psi^\infty_n : \check{H}^\infty_n(X; G) \to \check{H}^\infty_n(X; G)_{(X,A)}$$

and

$$\Psi^n_\infty : \hat{H}^n_\infty(X; G)^{(X,A)} \to \hat{H}^n_\infty(X; G)$$

are isomorphisms.

Now consider the diagrams

$$\check{H}_n^{\infty}(X,A;G) \xrightarrow{\partial_n'} \check{H}_{n-1}^{\infty}(A;G)_{(X,A)} \xleftarrow{\Phi_{n-1}^{\infty}} \check{H}_{n-1}^{\infty}(A;G)$$

and

$$\hat{H}_{\infty}^{n}(A;G) \xleftarrow{\Phi_{\infty}^{n}} \hat{H}_{\infty}^{n}(A;G)^{(X,A)} \xrightarrow{\delta'^{n}} \hat{H}_{\infty}^{n+1}(X,A;G)$$

and define the boundary homomorphism of Čech border homology groups and cobaundary homomorphism of Čech border cohomology groups as compositions

$$\partial_n^{\infty} = (\Phi_{n-1}^{\infty})^{-1} \cdot \partial_n'$$

and

$$\delta_{\infty}^{n} = \delta'^{n} \cdot (\Psi_{\infty}^{n})^{-1}.$$

Thus, we have obtained the following theorems.

Theorem 1.6. *Let $f : (X,A) \to (Y,B)$ be a proper map. Then hold the following equalities*

$$(f_{|A})_* \cdot \partial_n^{\infty} = \partial_n^{\infty} \cdot f_*$$

and

$$\delta_{\infty}^{n-1}(f_{|A})^* = f^* \cdot \delta_{\infty}^{n-1}.$$

Proof. The proof follows from the following commutative diagrams

$$\begin{tikzcd}
\check{H}_n^{\infty}(X,A;G) \arrow[r, "\partial_n'"] \arrow[d, "f_*"'] & \check{H}_{n-1}^{\infty}(A;G)_{(X,A)} \arrow[d, "(f_{|A})_*'"'] & \check{H}_{n-1}^{\infty}(A;G) \arrow[l, "\Phi_{n-1}^{\infty}"'] \arrow[d, "(f_{|A})_*"] \\
\check{H}_n^{\infty}(Y,B;G) \arrow[r, "\partial_n'"'] & \check{H}_{n-1}^{\infty}(B;G)_{(Y,B)} & \check{H}_{n-1}^{\infty}(B;G) \arrow[l, "\Phi_{n-1}^{\infty}"]
\end{tikzcd}$$

and

$$\begin{tikzcd}
\hat{H}_{\infty}^{n-1}(B;G) \arrow[r, "\Phi_{\infty}^{n-1}"] \arrow[d, "(f_{|A})^*"'] & \hat{H}_{\infty}^{n-1}(B;G)_{(Y,B)} \arrow[r, "\delta'^{n}"] \arrow[d, "(f_{|A})'^*"'] & \hat{H}_{\infty}^{n}(Y,B;G) \arrow[d, "f^*"] \\
\hat{H}_{\infty}^{n-1}(A;G) \arrow[r, "\Phi_{\infty}^{n-1}"'] & \hat{H}_{\infty}^{n-1}(A;G)_{(X,A)} \arrow[r, "\delta'^{n}"'] & \hat{H}_{\infty}^{n}(X,A;G),
\end{tikzcd}$$

where $(f_{|A})'_*$ and $(f_{|A})^{*'}$ are defined as the appropriate limit homomorphisms. $\square$

Let $i : A \to X$ and $j : X \to (X, A)$ be the inclusion maps.

Theorem 1.7. *Let $(X, A) \in ob(\mathscr{N}^2)$. Then the Čech border cohomology sequence*

$$\cdots \to H_\infty^{n-1}(A; G) \xrightarrow{\delta_\infty^{n-1}} H_\infty^n(X, A; G) \xrightarrow{j^*} H_\infty^n(X; G) \xrightarrow{i^*} H_\infty^n(A; G) \longrightarrow \cdots$$

is exact while the Čech border homology sequence

$$\cdots \leftarrow H^\infty_{n-1}(A; G) \xleftarrow{\partial^\infty_{n-1}} H^\infty_n(X, A; G) \xleftarrow{j_*} H^\infty_n(X; G) \xleftarrow{i_*} H^\infty_n(A; G) \longleftarrow \cdots$$

is a partially exact.

Proof. This theorem we can prove analogously to the corresponding theorem of the classical Čech theory [E-St]. $\square$

Theorem 1.8. *Let $(X, A) \in ob(\mathscr{M}^2)$ and G be an abelian group. If $\mathscr{U}$ is open in X and $\bar{U} \subset \mathrm{int}A$, then the inclusion map $i : (X \setminus U, A \setminus U) \to (X, A)$ induces isomorphisms*

$$i_* : \check{H}^\infty_n(X \setminus U, A \setminus U) \to \check{H}^\infty_n(X, A; G)$$

and

$$j^* : \hat{H}^n_\infty(X, A; G) \to \hat{H}^n_\infty(X \setminus U, A \setminus U)$$

Proof. Let $\mathrm{cov}'_\infty(X, A)$ be the subset of $\mathrm{cov}(X, A)$ consisting of all coverings $\alpha = \{\alpha_v\}_{v \in V_\alpha, V_\alpha^A}$ with property:

if $\alpha_v \cap U \neq \emptyset$, then $v \in V_\alpha^A$ and $\alpha_v \subset A$.

First prove that $\mathrm{cov}'_\infty(X, A)$ is cofinal in $\mathrm{cov}_\infty(X, A)$. Let $\alpha = \{\alpha_v\}_{v \in (V_\alpha, V_\alpha^A)}$ be a border covering of (X, A) with enclosure K_α. Let V' be a set such that $V' \cap V_\alpha = \emptyset$ and there exists a bijective function between V_α^A and V'. Let $v \in V_\alpha^A$. The correspondence element of v in V' denote by v'. Now define the border covering $\gamma = \{\gamma_v\}_{v \in (V_\alpha \cup V', V_\alpha^A \cup V')} \in \mathrm{cov}_\infty(X, A)$. Let

$$\gamma_v = \alpha_v \setminus \bar{U}, \quad v \in V_\alpha$$

and

$$\gamma_{v'} = \alpha_v \cap \mathrm{int} A, \qquad v^{'} \in V^{'}.$$

It is clear that γ is a border covering of (X, A) with enclosure K_α and $\gamma \geq \alpha$.

Now prove that $i^{-1}(\mathrm{cov}^{'}_{\infty}(X, A))$ is cofinal in $\mathrm{cov}_{\infty}(X \setminus U, A \setminus U)$. Let $\beta = \{\beta_v\}_{v \in (V_\beta, V_\beta^{A \setminus U})}$ be a border covering of $(X \setminus U, A \setminus U)$ with enclosure K_β. Define a border covering $\alpha = \{\alpha_v\}_{v \in (V_\beta, V_\beta^{A \setminus U})} \in \mathrm{cov}_{\infty}(X, A)$.

Let

$$\alpha_v = \beta_v \cup U.$$

The family $\alpha = \{\alpha_v\}_{v \in (V_\beta, V_\beta^{A \setminus U})}$ is a border covering of (X, A) with enclosure K_β.

Let $\gamma \in \mathrm{cov}^{'}_{\infty}(X, A)$ be a border covering such that $\gamma \geq \alpha$. It is clear that $i^{-1}(\gamma) \geq \beta = i^{-1}(\alpha)$.

As in [E-St] we can prove that there exist isomorphisms

$$i_{\alpha *} : H_n((X \setminus U)_\beta, (A \setminus U)_\beta; G) \to H_n(X_\alpha, A_\alpha; G)$$

and

$$i^*_\alpha : H^n(X_\alpha, A_\alpha; G) \to H^n((X \setminus U)_\beta, (A \setminus U)_\beta; G).$$

The conclusion of the theorem is a consequence of these isomorphisms. $\square$

Theorem 1.9. *If X is a compact space, then for each $n \neq 0$*

$$\check{H}^{\infty}_n(X; G) = 0 = \hat{H}^n_{\infty}(X; G)$$

and

$$\check{H}^{\infty}_0(X; G) = G = \hat{H}^0_{\infty}(X; G).$$

Proof. Let $\alpha \in \mathrm{cov}_{\infty}(X)$ be the border covering of X consisting of empty set. It is clear that α is a refinement of any border covering of X. The set $\{\alpha\}$ is a cofinal subset of $\mathrm{cov}_{\infty}(X)$. Consider the inverse system $\{H_n(X_\alpha; G), p^{\alpha}_{\alpha *}, \{\alpha\}\}$ and direct system $\{H^n(X_\alpha; G), p^{\alpha *}_{\alpha}, \{\alpha\}\}$. Note that

$$\varprojlim \{H_n(X_\alpha; G), p^{\alpha}_{\alpha *}, \{\alpha\}\} = \check{H}^{\infty}_n(X; G) = H_n(X_\alpha; G)$$

and

$$\varinjlim \{H^n(X_\alpha; G), p^{\alpha *}_{\alpha}, \{\alpha\}\} = \check{H}^n_{\infty}(X; G) = H^n(X_\alpha; G).$$

The nerve X_α consists of one vertex. Using the methods of proofs of results VI.3.8 and VI.4.3 of [E-St] we can conclude that

$$\check{H}_n^\infty(X;G) = 0 = \hat{H}_\infty^n(X;G)$$

and

$$\hat{H}_0^\infty(X;G) = G = \hat{H}_\infty^0(X;G).$$

$\square$

Thus, $(\hat{H}_\infty^n(-,-;G))$ $(\check{H}_n^\infty(-,-;G)) : \mathscr{N}^2 \to \mathscr{A}b$ Čech border (co)homology functors satisfy the Steenrod-Eilenberd type axioms (cf.[E-St]): Axiom of natural transformation, (axiom of exactness) axiom of partially exactness, axiom of excision and axiom of dimension, but they don't satisfy the proper homotopy axiom.

2 On some applications of Čech border homology and cohomology groups

In the section all spaces under discussion are metrizable. We are mainly interested in the following problem: how can be characterized the Čech homology and cohomology groups, coefficient of cylicities and cohomological dimensions of remainders of Stone-Čech compactifications.

The main result about the connection between Čech (co)homology groups of remainders and Čech border (co)homology groups of spaces is incorporated in the following theorem.

Theorem 2.1. *Let $(X,A) \in ob(\mathscr{M}^2)$ and $(\beta X, \beta A)$ its Stone-Čech compacification. Then*

$$\check{H}_n^f(\beta X \setminus X, \beta A \setminus A; G) = \check{H}_n^\infty(X,A;G)$$

and

$$\hat{H}_f^n(\beta X \setminus X, \beta A \setminus A; G) = \hat{H}_\infty^n(X,A;G).$$

Proof. Let $\alpha = \{\alpha_v\}_{v\in(V_\alpha, V_\alpha^{\beta A\setminus A})}$ and $\alpha^{'} = \{\alpha^{'}_w\}_{w\in(W_{\alpha^{'}}, W_{\alpha^{'}}^{\beta A\setminus A})}$ be the closed covers of pairs $(\beta X \setminus X, \beta A \setminus A)$ and $\alpha \geq \alpha^{'}$. By Lemma 4 of [Sm_4] there exist open in βX swelling $\beta_1 = \{\beta^1_v\}_{v\in(V_\alpha,V_\alpha^{\beta A\setminus A})}$ and $\beta' = \{\beta'_w\}_{w\in(W_{\alpha'},W_{\alpha'}^{\beta A\setminus A})}$ of α and α', respectively. Assume that $\alpha_v \subseteq \alpha'_{w_k}$, $k = 1, 2, \cdots, m_v$. Let

$$\beta_v = \beta^1_v \cap (\bigcap_{k=1}^{m_v} \beta'_{w_k}), \qquad v \in V_\alpha.$$

Note that $\alpha_v \subset \beta_v \subset \beta^1_v$ for each $v \in V_\alpha$. It is clear that $\beta = \{\beta_v\}_{v\in(V_\alpha,V_\alpha^A)}$ is a swelling of $\alpha = \{\alpha_v\}_{v\in(V_\alpha,V_\alpha^{\beta A\setminus A})}$ and $\beta \geq \beta'$.

The swelling in βX of closed cover α of $(\beta X \setminus X, \beta A \setminus A)$ denote by $s(\alpha)$. Let S be the set of all such type swellings.

Now define an order $\geq'$ in S. By definition,

$$s(\alpha') \geq' s(\alpha) \Leftrightarrow s(\alpha') \geq s(\alpha) \ \wedge \ \alpha' \geq \alpha.$$

It is clear that S is directed by $\geq'$. Let $((\beta X \setminus X)_{s(\alpha)}, (\beta A \setminus A)_{s(\alpha)})$ be the nerve of $s(\alpha) \in S$ and $p_{s(\alpha)s(\alpha')}$ be the projection simplicial map induced by the refinement $\alpha' \geq \alpha$. Consider an inverse system

$$\{H_n((\beta X \setminus X)_{s(\alpha)}, (\beta A \setminus A)_{s(\alpha)}; G), p^{s(\alpha')}_{s(\alpha)*}, S\}$$

and a direct system

$$\{H^n((\beta X \setminus X)_{s(\alpha)}, (\beta A \setminus A)_{s(\alpha)}; G), p^{s(\alpha')*}_{s(\alpha)}, S\}.$$

Let $\varphi : S \to \mathrm{cov}^{\mathrm{cl}}_{\mathrm{f}}(\beta \mathrm{X} \setminus \mathrm{X}, \beta \mathrm{A} \setminus \mathrm{A})$ be a function in the set of closed finite covers of pair $(\beta X \setminus X, \beta A \setminus A)$ given by formula

$$\varphi(s(\alpha)) = \alpha, \qquad s(\alpha) \in S.$$

Note that φ is an increasing function and

$$\varphi(S) = \mathrm{cov}^{\mathrm{cl}}_{\mathrm{f}}(\beta \mathrm{X} \setminus \mathrm{X}, \beta \mathrm{A} \setminus \mathrm{A}).$$

For each index $s(\alpha) \in S$ we have

$$H_n((\beta X \setminus X)_{s(\alpha)}, (\beta A \setminus A)_{s(\alpha)}; G) = H_n((\beta X \setminus X)_\alpha, (\beta A \setminus A)_\alpha; G)$$

and

$$H^n((\beta X \setminus X)_{s(\alpha)}, (\beta A \setminus A)_{s(\alpha)}; G) = H^n((\beta X \setminus X)_\alpha, (\beta A \setminus A)_\alpha; G).$$

It is known that for normal spaces the Čech (co)homology groups based on finite open covers and finite closed covers are isomorphical. By Theorems 3.14 and 4.13 of ([E-St],Ch.VIII) we have

$$H_n^f(\beta X\backslash X, \beta A\backslash A; G) \approx \lim_{\longleftarrow}\{H_n((\beta X\backslash X)_{s(\alpha)}, (\beta A\backslash A)_{s(\alpha)}; G), p_{s(\alpha)*}^{s(\alpha')}, S\} \quad (1)$$

and

$$H_f^n(\beta X \setminus X, \beta A \setminus A; G) \approx \lim_{\longrightarrow}\{H^n((\beta X\backslash X)_{s(\alpha)}, (\beta A\backslash A)_{s(\alpha)}; G), p_{s(\alpha)}^{s(\alpha')*}, S\} \quad (2)$$

For each swelling $s(\alpha) = \{s(\alpha)_v\}_{v\in(V_\alpha, V_\alpha^{\beta A\setminus A})} \in S$ the family

$$s(\alpha) \wedge X = \{s(\alpha)_v \cap X\}_{v\in(V_\alpha, V_\alpha^{\beta A\setminus A})}$$

is a border cover of (X, A).

Let $\psi : S \to \mathrm{cov}_\infty(X, A)$ be the function defined by formula

$$\psi(s(\alpha)) = s(\alpha) \wedge X, \quad s(\alpha) \in S.$$

The function ψ is increases and $\psi(S)$ is a cofinal subset of $\mathrm{cov}_\infty(X, A)$. Note that the correspondence

$$((\beta X\backslash X)_{s(\alpha)}, (\beta A\backslash A)_{s(\alpha)}) \to (X_{s(\alpha)\wedge X}, A_{s(\alpha)\wedge X}) : s(\alpha)_v \to s(\alpha)_v \cap X, \quad v \in V_\alpha$$

induces an isomorphism of pairs of simplicial complexes. Thus, for each $s(\alpha) \in S$ we have the isomorphisms

$$H_n((\beta X \setminus X)_{s(\alpha)}, (\beta A \setminus A)_{s(\alpha)}; G) = H_n(X_{s(\alpha)\wedge X}, A_{s(\alpha)\wedge X}; G)$$

and

$$H^n((\beta X \setminus X)_{s(\alpha)}, (\beta A \setminus A)_{s(\alpha)}; G) = H^n(X_{s(\alpha)\wedge X}, A_{s(\alpha)\wedge X}; G).$$

By Theorems 3.15 and 4.13 of ([E-St],ch.VIII)

$$\check{H}_n^\infty(X, A; G) = \lim_{\longleftarrow}\{H_n((\beta X\backslash X)_{s(\alpha)}, (\beta A\backslash A)_{s(\alpha)}; G), p_{s(\alpha)*}^{s(\alpha')}, S\} \quad (3)$$

and

$$\hat{H}_\infty^n(X, A; G) = \lim_{\longrightarrow}\{H_n((\beta X\backslash X, \beta A\backslash A)_{s(\alpha)}; G), p_{s(\alpha)}^{s(\alpha')*}, S\}. \quad (4)$$

From (1), (2), (3) and (4) it follows that

$$\check{H}_n^{\infty}(X,A;G) = \check{H}_n^{f}(\beta X \setminus X, \beta A \setminus A; G)$$

and

$$\hat{H}^n_{\infty}(X,A;G) = \hat{H}^n_f(\beta X \setminus X, \beta A \setminus A; G).$$

□

Now give the following definitions and results.

Definition 2.2. Let G be an abelian group and n nonnegative integer. A border (co)homological coefficient of cyclisity of pair $(X,A) \in ob(\mathscr{M}^2)$ with respect to G denoted by $(\eta_G^{\infty}(X,A))$ $\eta^G_{\infty}(X,A)$ is n, if $(\hat{H}^m_{\infty}(X,A;G) = 0)$ $\check{H}_m^{\infty}(X,A;G) = 0$ for all $m > n$ and $(\hat{H}^n_{\infty}(X,A;G) \neq 0)$ $\check{H}_n^{\infty}(X,A;G) \neq 0$.

$(\eta_G^{\infty}(X,A) = +\infty)$ $\eta^G_{\infty}(X,A) = +\infty$ if for every m there is $n \geq m$ with $(\hat{H}^n_{\infty}(X,A;G) \neq 0)$ $\check{H}_n^{\infty}(X,A;G) \neq 0$.

Analogously are defined the (co)homological coefficient of cyclicity $(\eta_G(X,A))$ $\eta^G(X,A)$ of pair (X,A) (cf. [Bo], [No]).

Theorem 2.3. *For each pair $(X,A) \in ob(\mathscr{M}^2)$*

$$\eta_G^{\infty}(X,A) = \eta_G(\beta X \setminus X, \beta A \setminus A)$$

and

$$\eta^G_{\infty}(X,A) = \eta^G(\beta X \setminus X, \beta A \setminus A).$$

Proof. This is an immediate consequence of theorem 2.1. Indeed, Let $\eta_G(\beta X \setminus X, \beta A \setminus A) = n$. Then for each $m > n$, $\hat{H}^m_f(\beta X \setminus X, \beta A \setminus A; G) = 0$ and $\hat{H}^n_f(\beta X \setminus X, \beta A \setminus A; G) \neq 0$. From the isomorphism

$$\hat{H}^k_f(\beta X \setminus X, \beta A \setminus A; G) = \hat{H}^k_f(X,A;G)$$

it follows that $\hat{H}^m_{\infty}(X,A;G) = 0$ for each $m > n$ and $\hat{H}^n_{\infty}(X,A;G) \neq 0$. Thus, $\eta_G^{\infty}(X,A) = n = \eta_G(\beta X \setminus X, \beta A \setminus A)$.

Analogously we can prove equality $\eta^G_{\infty}(X,A) = \eta^G(\beta X \setminus X, \beta A \setminus A)$. □

Definition 2.4. A border small cohomological dimension $d^f_{\infty}(X;G)$ with respect G is defined to be the smallest integer n such that, whenever $m \geq n$ and A is closed in X, the homomorphism $i^* : \hat{H}^m_{\infty}(X;G) \to \hat{\mathrm{H}}^m_{\infty}(A;G)$ induced by the inclusion $i : A \to X$ is an epimorphism.

Theorem 2.5. *Let $X \in ob(\mathscr{M})$. Then the following relation*

$$d^f_\infty(X;G) \leq d_f(\beta X \setminus X;G)$$

hold, where $d_f(\beta X \setminus X;G)$ is a small cohomological dimension of $\beta X \setminus X$ (see [N], p.199*).*

Proof. Let A be a closed subset of X. Assume that $d^f(\beta X \setminus X;G) = n$. Then for each $m \geq n$ the homomorphism $i^*_{\beta X\setminus X} : \hat{H}^m_f(\beta X\setminus X;G) \to \hat{H}^m_f(\beta A\setminus A;G)$ is an epimorphim. Consider the following commutative diagram

$$\begin{array}{ccc}
\hat{\mathrm{H}}^m_\infty(X;G) & \approx & \hat{\mathrm{H}}^m_f(\beta X \setminus X;G) \\
{\scriptstyle i^*_A}\downarrow & & \downarrow{\scriptstyle i^*_{\beta A\setminus A}} \\
\hat{\mathrm{H}}^m_\infty(A;G) & \approx & \hat{\mathrm{H}}^m_f(\beta A \setminus A;G).
\end{array}$$

It is clear that the homomorphim

$$i^*_A : \hat{H}^m_f(X;G) \to \hat{H}^m_f(A;G)$$

also is an epimorphim for each $m \geq n$. Thus, $d^f_\infty(X;G) \leq n = d^f(\beta X \setminus X;G)$. ☐

Proposition 2.6. Let $(X, A) \in ob(\mathscr{M}^2)$. Then

$$d^\infty_f(A;G) \leq d^\infty_f(X;G).$$

Proof. Let B be an arbitrary closed subset of A and $j : B \to A$, $i : A \to X$ and $k : B \to X$ be the inclusion maps. Note that $k = i{\cdot}j$. The induced homomorphism maps $k^* : \hat{H}^n_\infty(X;G) \to \hat{H}^n_\infty(B;G)$, $i^* : \hat{H}^n_\infty(X;G) \to \hat{H}^n_\infty(A;G)$ and $j^* : \hat{H}^n_\infty(A;G) \to \hat{H}^n_\infty(B;G)$ satisfy the relation $k^* = j^* \cdot i^*$.

Let $n = d^\infty_f(X;G)$. For each $m \geq n$ the homomorphisms $k^* : \hat{H}^m_\infty(X;G) \to \hat{H}^m_\infty(B;G)$ and $i^* : \hat{H}^m_\infty(X;G) \to \hat{H}^m_\infty(A;G)$ are epimorphisms. Hence, $j^* : \hat{H}^m_\infty(A;G) \to \hat{H}^m_\infty(B;G)$ homomorphism also is an ephimorphism for each $m \geq n$. Thus, $d^\infty_f(A;G) \leq n = d^\infty_f(X;G)$. ☐

Corollary 2.7. *For each closed subspace Y of metrizable space X*

$$d^\infty_f(Y;G) \leq d_f(\beta X \setminus X;G).$$

Remark 2.8. The results of this paper also hold for spaces satisfying the compact axiom of countability. A space X satisfies the compact axiom of countability if for each compact subset $B \subset X$ there exist a compact subset $B' \subset X$ such that $B \subset B'$ and B' has a countable of finite fundamental system of neighbourhood (see Definition 4 of [Sm$_4$], p.143). A space X is complete in the sense of Čech if and only if it is G_δ type set in some compact extension. Each locally metrizable spaces, complete in the seance Čech spaces [Č] and locally compact spaces satisfy the compact axiom of countability.

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