$\mathfrak{G}$-bases in free objects of Topological Algebra
(Local) ω^ω-bases in topological and uniform spaces

Taras Banakh and Arkady Leiderman

Lviv & Kielce

Prague, 29 July 2016

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(Local) Bases indexed by posets

Let P be a poset, i.e., a set endowed with a partial order $\leq$.

Definition (Topological)

A topological space X has a *local P-base* at a point $x \in X$ if X has a neighborhood base $(U_\alpha)_{\alpha\in P}$ at x such that $U_\beta \subset U_\alpha$ for any $\alpha \leq \beta$ in P.
A topological space X has a *local P-base* if X has a local P-base at each point $x \in X$.

Definition (Uniform)

A uniform space X has a *P-base* (or is *P-based*) if its uniformity $\mathcal{U}(X)$ has a base $\{U_\alpha\}_{\alpha\in P}$ such that $U_\beta \subset U_\alpha$ for all $\alpha \leq \beta$ in P.

Example

A topological space X has a local ω-base $\Leftrightarrow$ X is first-countable.
A uniform space X has an ω-base $\Leftrightarrow$ X is metrizable.

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(Local) P-bases for countable posets

Let P be a countable poset or, more generally, a poset containing a countable cofinal subset.

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A topological space X has a local P-base $\Leftrightarrow$ X is first-countable.
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So, for countable posets P (local) P-bases give nothing new.
One of the simplest posets of uncountable cofinality is the countable power ω^ω of the countable cardinal ω, endowed with the partial order $\le$ defined by $f \le g$ iff $f(n) \le g(n)$ for all $n \in \omega$.

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For the poset ω^ω, topological spaces with local ω^ω-base are called spaces with a $\mathfrak{G}$-base.

This terminology came from Functional Analysis and was brought to Topological Algebra and General Topology by Jerzy Kąkol.

But we prefer and agitate to use the more self-suggesting terminology of local ω^ω-bases for topological spaces and ω^ω-bases for uniform spaces.

Our **Initial Problem** was: Characterize topological spaces whose free objects (like free topological groups or free locally convex spaces) have a local ω^ω-base.

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Stability properties of the class of topological spaces with a local ω^ω-base

Theorem

The class of topological spaces X with a local ω^ω-base contains all first-countable spaces and is stable under taking

- *subspaces,*
- *images under open maps,*
- *countable Tychonoff products,*
- *countable box-products,*
- *inductive topologies determined by countable covers,*
- *images under pseudo-open maps with countable fibers.*

Corollary

Each submetrizable k_ω-space has a local ω^ω-base (since any such space embeds into the countable box-power of the Hilbert cube).

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Character of spaces with a local ω^ω-base

Theorem

If a topological space X has a local ω^ω-base at a point $x \in X$, then at this point the space X has character $\chi(x; X) \in \{1, \omega\} \cup [\mathfrak{b}, \mathfrak{d}]$.

Example

For a cardinal $\kappa \in \{\mathfrak{b}, \mathfrak{d}, \mathrm{cf}(\mathfrak{d})\}$ the ordinal segment $[0, \kappa]$ has a local ω^ω-base at the point κ.

Proof.

For $\kappa = \mathfrak{b}$, choose an unbounded subset $\{x_\alpha\}_{\alpha \in \mathfrak{b}} \subset \omega^\omega$ in the poset $(\omega^\omega, \le^*)$ and define an ω^ω-base $(U_x)_{x \in \omega^\omega}$ at $\mathfrak{b} \in [0, \mathfrak{b}]$ by $U_x = (\alpha_x, \mathfrak{b}]$ where $\alpha_x = \min\{\alpha \in \mathfrak{b} : x_\alpha \not\le^* x\}$.
For $\kappa = \mathfrak{d}$ choose a dominating set $\{x_\alpha\}_{\alpha \in \mathfrak{d}}$ in the poset ω^ω and define an ω^ω-base $(U_x)_{x \in \omega^\omega}$ at $\mathfrak{d} = [0, \mathfrak{d}]$ by $U_x = (\alpha_x, \mathfrak{d}]$ where $\alpha_x = \min\{\alpha \in \mathfrak{d} : x \le^* x_\alpha\}$. □

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Compact spaces with a (local) ω^ω-base

Example

Under $\omega_1 = \mathfrak{b}$ the ordinal segment $[0, \omega_1]$ has a local ω^ω-base.
Under $\omega_1 = \mathfrak{b} < \mathfrak{d} = \omega_2$ the segment $[0, \omega_2]$ has a local ω^ω-base.

According to a famous theorem of Arhangel'skii, each first-countable compact Hausdorff space has cardinality $\leq \mathfrak{c}$.

Problem

Is $|X| \leq \mathfrak{c}$ for any compact Hausdorff space X with a local ω^ω-base?

Theorem (Cascales-Orihuela, 1987)

Each compact ω^ω-based uniform space is metrizable.

What can be said about non-compact ω^ω-based uniform spaces?
Informal answer: Such spaces have many features of generalized metric spaces.

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A family $\mathcal{N}$ of subsets of a topological space X is called

- a *network* at $x \in X$ if for every neighborhood $O_x \subset X$ of x there is a set $N \in \mathcal{N}$ such that $x \in N \subset U$;
- a *cs^*-network* at x if for every neighborhood O_x of x and sequence $(x_n)_{n\in\omega}$ converging to x there is a set $N \in \mathcal{N}$ such that $x \in N \subset O_x$ and N contains infinitely many points x_n;
- a *Pytkeev*-network* at x if for every neighborhood O_x of x and sequence $(x_n)_{n\in\omega}$ accumulating at x there is $N \in \mathcal{N}$ such that $x \in N \subset O_x$ and N contains infinitely many points x_n.

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A topological space X is

- *strong Fréchet* at $x \in X$ if for any decreasing sequence $(A_n)_{n\in\omega}$ of subsets of X with $x \in \bigcap_{n\in\omega} \bar{A}_n$ there exists a sequence $(x_n)_{n\in\omega} \in \prod_{n\in\omega} A_n$ converging to x;
- *countable fan tightness* at $x \in X$ if for any decreasing sequence $(A_n)_{n\in\omega}$ of subsets of X with $x \in \bigcap_{n\in\omega} \bar{A}_n$ there exists a sequence $(F_n)_{n\in\omega}$ of finite subsets $F_n \subset A_n$ such that each neighborhood of x meets infinitely many sets F_n.

Proposition (folklore)

For a topological space X and a point $x \in X$ TFAE:

1. *X has a countable neighborhood base at x.*
2. *X has a countable cs*-network at x and is strong Fréchet at x.*
3. *X has a countable Pytkeev* network at x and has countable fan tightness at x.*

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A topological space X is

- *strong Fréchet* at $x \in X$ if for any decreasing sequence $(A_n)_{n\in\omega}$ of subsets of X with $x \in \bigcap_{n\in\omega} \bar{A}_n$ there exists a sequence $(x_n)_{n\in\omega} \in \prod_{n\in\omega} A_n$ converging to x;
- *countable fan tightness* at $x \in X$ if for any decreasing sequence $(A_n)_{n\in\omega}$ of subsets of X with $x \in \bigcap_{n\in\omega} \bar{A}_n$ there exists a sequence $(F_n)_{n\in\omega}$ of finite subsets $F_n \subset A_n$ such that each neighborhood of x meets infinitely many sets F_n.

Proposition (folklore)

For a topological space X and a point $x \in X$ TFAE:

1. *X has a countable neighborhood base at x.*
2. *X has a countable cs*-network at x and is strong Fréchet at x.*
3. *X has a countable Pytkeev* network at x and has countable fan tightness at x.*

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Spaces with a local ω^ω-base have a countable Pytkeev* network

Theorem (B., 2016)

If a topological space X has a local ω^ω-base at a point $x \in X$, then X has a countable Pytkeev* network at x.

Idea of the proof: Let $(U_\alpha)_{\alpha\in\omega^\omega}$ be a local ω^ω-base at x. Given a subset $A \subset \omega^\omega$ consider the intersection $U_A = \bigcap_{\alpha\in A} U_\alpha$. Let $\omega^{<\omega} = \bigcup_{n\in\omega} \omega^n$ and for every $\beta \in \omega^n \subset \omega^{<\omega}$ consider the basic clopen set ${\uparrow}\beta = \{\alpha \in \omega^\omega : \alpha|n = \beta\} \subset \omega^\omega$.

Lemma

The countable family $(U_{\uparrow\beta})_{\beta\in\omega^{<\omega}}$ is a Pytkeev* network at x.

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Proof of the lemma

Lemma

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Idea of the proof: Given a sequence $(x_n)_{n\in\omega}$ accumulating at x, use the ω^ω-base $(U_\alpha)_{\alpha\in\omega^\omega}$ to prove that the filter

$$\mathcal{F} = \big\{\{n\in\omega : x_n \in O_x\} : O_x \text{ is a neighborhood of } x\big\}$$

is analytic as a subset of $\mathcal{P}(\omega)$ and hence is meager. Then apply the Talagrand characterization of meager filters to find a finite-to-one map $\varphi:\omega\to\omega$ such that $\varphi(\mathcal{F})$ is a Fréchet filter. This map φ can be used to prove that for every $\alpha\in\omega^\omega$ there exists $k\in\omega$ such that $U_{\uparrow(\alpha|k)}$ contains infinitely many points x_n, $n\in\omega$.

The cardinality of spaces with a local ω^ω-base

Theorem (B.-Zdomskyy, 27.07.2016)

If a countably tight space X has a countable Pytkeev* network at any point, then $|X| \le 2^{L(X)}$ where $L(X)$ is the Lindelöf number of X.

Corollary (B.-Zdomskyy, 27.07.2016)

Each countably tight space X with a local ω^ω-base has $|X| \le 2^{L(X)}$.

Example

For any cardinal κ the ordinal segment $[0, \kappa]$ has a countable Pytkeev* network at each point.

Problem

Is $|X| \le \mathfrak{c}$ for any compact Hausdorff space X with a local ω^ω-base?

The answer is "yes" if $2^{\mathfrak{d}} = \mathfrak{c}$.

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Theorem

A topological space X is first countable at $x \in X$ if and only if X has a local ω^ω-base at x and X has countable fan tightness at x.

A subset A of a topological space X is called a *$\bar{G}_\delta$-set* if

$$A = \bigcap_{n\in\omega} U_n = \bigcap_{n\in\omega} \bar{U}_n$$

for some sequence $(U_n)_{n\in\omega}$ of open sets.

A subset of a normal space is $\bar{G}_\delta$ if and only if it is G_δ.

The following metrization theorem follows from the Metrization Theorem of Moore.

Theorem

A topological space X is metrizable if and only if X is first-countable, each closed subset of X is a $\bar{G}_\delta$-set in X and the topology of X is generated by an ω^ω-based uniformity.

Corollary

A topological space X is metrizable and separable if and only if X is first-countable, hereditarily Lindelöf and the topology of X is generated by an ω^ω-based uniformity.

Metrizability versus ω^ω-base of the uniformity

A subset A of a topological space X is called a *$\bar{G}_\delta$-set* if

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First-countablity of ω^ω-based uniform spaces

Theorem

For an ω^ω-based uniform space X the following conditions are equivalent:

1. *X is first-countable at x;*
2. *X has countable fan tightness at x;*
3. *X is a q-space at x.*

A topological space X is called a *q-space at* $x \in X$ if there are neighborhoods $(U_n)_{n\in\omega}$ of x such that each sequence $(x_n)_{n\in\omega} \in \prod_{n\in\omega} U_n$ has an accumulation point x_∞ in X.

Definition

A topological space X is called

1. a space with a *G_δ-diagonal* if the diagonal of the square $X \times X$ is a G_δ-set in X; this happens if and only if there exists a sequence $(\mathcal{U}_n)_{n\in\omega}$ of open covers of X such that $\{x\} = \bigcap_{n\in\omega} \mathcal{S}t(x, \mathcal{U}_n)$ for each $x \in X$;
2. a *$w\Delta$-space* if there exists a sequence $(\mathcal{U}_n)_{n\in\omega}$ of open covers of X such that for every $x \in X$, any sequence $(x_n)_{n\in\omega} \in \prod_{n\in\omega} \mathcal{S}t(x, \mathcal{U}_n)$ has an accumulation point in X;
3. an *M-space* if there exists a sequence $(\mathcal{U}_n)_{n\in\omega}$ of open covers of X such that each $\mathcal{U}_{n+1}$ star-refines $\mathcal{U}_n$ and for every $x \in X$, any sequence $(x_n)_{n\in\omega} \in \prod_{n\in\omega} \mathcal{S}t(x, \mathcal{U}_n)$ has an accumulation point in X.

metrizable $\Leftrightarrow$ M-space with a G_δ-diagonal

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$w\Delta$-spaces, M-spaces and G_δ-diagonals

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metrizable $\Leftrightarrow$ M-space with a G_δ-diagonal

Theorem

A topological space X has a G_δ-diagonal if X is a $w\Delta$-space and the topology of X is generated by an ω^ω-based uniformity.

Corollary

A topological space X is metrizable if and only if X is an M-space and the topology of X is generated by an ω^ω-based uniformity.

Corollary (Cascales-Orihuela)

A compact space is metrizable if and only if its topology is generated by an ω^ω-based uniformity.

ω^ω-based uniform $w\Delta$-spaces have a G_δ-diagonal

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Definition

A family $\mathcal{N}$ of subsets of a topological space X is called

- a *network* if for each point $x \in X$ and neighborhood $O_x \subset X$ of x there is a set $N \in \mathcal{N}$ such that $x \in N \subset O_x$;
- a *$\mathcal{C}$-network* for a family $\mathcal{C}$ of subsets of X if for each set $C \in \mathcal{C}$ and neighborhood $O_C \subset X$ of C there is a set $N \in \mathcal{N}$ such that $C \subset N \subset O_x$.

Definition

A regular topological space X is called

- *cosmic* if X has a countable network;
- a *σ-space* if X has a σ-discrete network;
- a *Σ-space* if X has a σ-discrete $\mathcal{C}$-network for some family $\mathcal{C}$ of closed countably compact subsets of X.

Σ-space $\Rightarrow$ σ-space $\Rightarrow$ cosmic.

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Σ-spaces and σ-spaces

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ω^ω-based uniform Σ-spaces are σ-spaces

$$\Sigma\text{-space} \Rightarrow \sigma\text{-space} \Rightarrow \text{cosmic.}$$

Theorem

An ω^ω-based uniform space X is a Σ-space iff X is a σ-space.

Corollary (Cascales-Orihuela)

Each compact ω^ω-based uniform space is metrizable.

$$\Sigma\text{-space} \Rightarrow \sigma\text{-space} \Rightarrow \text{cosmic}.$$

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$\aleph_0$-spaces, $\aleph$-spaces, $\mathfrak{P}_0$-spaces, $\mathfrak{P}^*$-spaces

Definition

A topological space X is called

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- an $\aleph$-*space* if X has a σ-discrete cs^*-network;
- a $\mathfrak{P}_0$-*space* if X has a countable Pytkeev* network;
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$$\begin{array}{ccccccccc}
\text{metrizable separable} & \Longrightarrow & \mathfrak{P}_0\text{-space} & \Longrightarrow & \aleph_0\text{-space} & \Longrightarrow & \text{cosmic} & \Longrightarrow & \text{Lindelöf } \Sigma\text{-space} \\
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For an ω^ω-based uniform space the following equivalences hold:

1. *σ-space $\Leftrightarrow$ Σ-space.*
2. *paracompact $\mathfrak{P}^*$-space $\Leftrightarrow$ collectionwise normal Σ-space.*

Problem

Is each ω^ω-based uniform Σ-space a $\mathfrak{P}^*$-space?

ω^ω-based uniform σ-spaces are $\mathfrak{P}^*$-spaces

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For a uniform space X by $\mathcal{U}(X)$ we denote the universality of X.

Definition

A function $f : X \to Y$ between uniform spaces is called *ω-continuous* if for every untourage $U \in \mathcal{U}(Y)$ there exists a countable subfamily $\mathcal{V} \subset \mathcal{U}(X)$ such that for every $x \in X$ there exists $V \in \mathcal{V}$ with $f(V[x]) \subset U[f(x)]$.
Here $V[x] = \{y \in X : (x, y) \in V\}$ is the V-ball centered at x.

For a uniform space X let $C_\omega(X)$ and $C_u(X)$ be the spaces of all ω-continuous and uniformly continuous real-valued functions on X, respectively.
It is clear that $C_u(X) \subset C_\omega(X) \subset C(X) \subset \mathbb{R}^X$.

If $\mathcal{U}(X)$ is the universal uniformity on a Tychonoff space X, then $C_u(X) = C_\omega(X) = C(X)$.

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Characterizing "small" ω^ω-based uniform spaces

Theorem

For an ω^ω-based uniform space X TFAE:

(1) *X contains a dense Σ-subspace with countable extent;*

(2) *X is separable;*

(3) *X is cosmic;*

(4) *X is an $\aleph_0$-space;*

(5) *X is a $\mathfrak{P}_0$-space.*

If $C_\omega(X) = C_u(X)$, then the conditions (1)–(5) are equivalent to:

(6) *X is σ-compact.*

(7) *$C_u(X)$ is cosmic (or analytic);*

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If $\omega_1 < \mathfrak{b}$, then (1)–(5) are equivalent to

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If $\omega_1 = \mathfrak{b}$, there exists a Lindelöf non-separable ω^ω-based space.

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A uniform space is ω-narrow if $width(X) \le \omega_1$, where

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If $\Delta_X \in \mathcal{U}(X)$, then the cardinal depth(X) is not defined.
In this case we put depth(X) $= \infty$ and assume that
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Local ω^ω-base in free objects of Topological Algebra

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The small uncountable cardinal $\mathfrak{e}^\sharp$

$$\mathfrak{e}^\sharp = \sup\{\kappa^+ : \omega \le \kappa = \mathrm{cf}(\kappa),\ \kappa^\kappa \le_T \omega^\omega\}$$

Theorem

$\mathfrak{e}^\sharp \in \{\omega_1\} \cup (\mathfrak{b}, \mathfrak{d}]$. *So,* $\mathfrak{b} = \mathfrak{d}$ *implies* $\mathfrak{e}^\sharp = \omega_1$.

Theorem (B., Zdomskyy)

1. *It is consistent that* $\mathfrak{b} < \mathfrak{d}$ *and* $\mathfrak{e}^\sharp = \omega_1$.
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Problem

Is $\mathfrak{e}^\sharp$ *equal to any known cardinal characteristic of the continuum?*

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References

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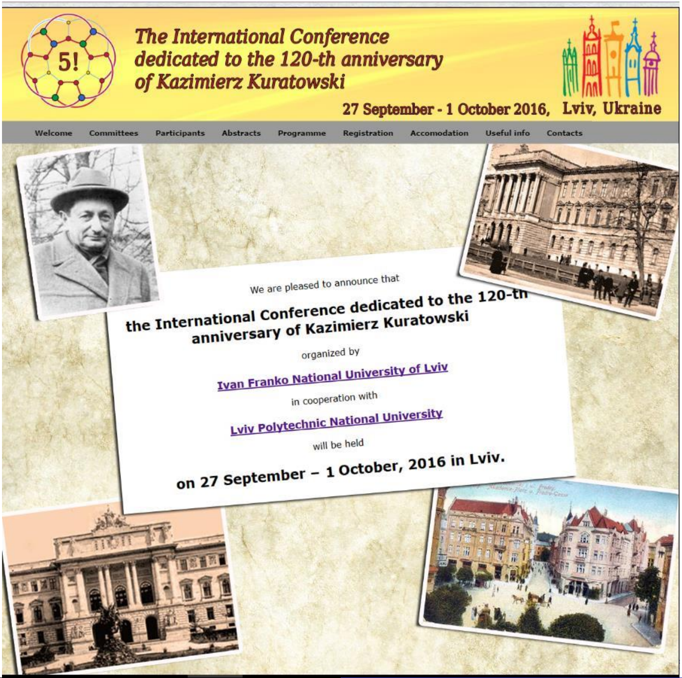
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The International Conference
dedicated to the 120-th anniversary
of Kazimierz Kuratowski
27 September - 1 October 2016, Lviv, Ukraine
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We are pleased to announce that
the International Conference dedicated to the 120-th anniversary of Kazimierz Kuratowski
organized by
Ivan Franko National University of Lviv
in cooperation with
Lviv Polytechnic National University
will be held
on 27 September – 1 October, 2016 in Lviv.