Tree sums and maximal connected I-spaces

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Twelfth Symposium on General Topology
Prague, July 2016

Maximal and minimal topologies

Definition

Let X be a set. The set of all topologies on X is a complete lattice denoted by $\mathcal{T}(X)$.
Let $\mathcal{P}$ be a property of topological spaces.

- We say a topology $\tau \in \mathcal{T}(X)$ is *maximal* $\mathcal{P}$ if it is a maximal element of $\{\sigma \in \mathcal{T}(X) : \sigma \text{ satisfies } \mathcal{P}\}$, i.e. τ satisfies $\mathcal{P}$ but no strictly finer topology satisfies $\mathcal{P}$. In that case $\langle X, \tau \rangle$ is a *maximal* $\mathcal{P}$ *space*.
- We say a topology $\tau \in \mathcal{T}(X)$ is *minimal* $\mathcal{P}$ if it satisfies $\mathcal{P}$ but no strictly coarser topology satisfies $\mathcal{P}$. In that case $\langle X, \tau \rangle$ is a *minimal* $\mathcal{P}$ *space*.

Maximal and minimal topologies

Examples

- *Maximal space* means *maximal without isolated points*.
- A compact Hausdorff space is both *maximal compact* and *minimal Hausdorff*.
- We are interested in *maximal connected spaces*.

For more examples see [Cameron, 1971].

Maximal connected topologies were first considered by Thomas in [Thomas, 1968]. Thomas proved that an open connected subspace of a maximal connected space is maximal connected, and characterized finitely generated maximal connected spaces.

Maximal connected spaces

Definition

A topological spaces is called

- *maximal connected* [Thomas, 1968] if it is connected and has no connected strict expansion;
- *strongly connected* [Cameron, 1971] if it has a maximal connected expansion;
- *essentially connected* [Guthrie–Stone, 1973] if it is connected and every connected expansion has the same connected subsets.

Observation

Every maximal connected space is both strongly connected and essentially connected.

Subspaces of maximal connected spaces

Lemma

Let $\langle Y, \sigma\rangle$ be a subspace of a connected space $\langle X, \tau\rangle$. For every connected expansion $\sigma^* \geq \sigma$ there exists a connected expansion $\tau^* \geq \tau$ such that $\tau^* \upharpoonright Y = \sigma^*$.

Sketch of the proof.
We put $\tau^* := \tau \vee \{S \cup (X \setminus \overline{Y}) : S \subseteq Y\ \sigma^*\text{-open}\}$.

Corollary

The following properties are preserved by connected subspaces:

- maximal connectedness [Guthrie–Reynolds–Stone, 1973],
- essential connectedness [Guthrie–Stone, 1973],
- strong & essential connectedness.

Strongly connected and essentially connected topologies

Theorem [Hildebrand, 1967]

The real line is essentially connected.

Theorem [Simon, 1978] and [Guthrie–Stone–Wage, 1978]

There exists a maximal connected expansion of the real line.

Corollary

The spaces $\mathbb{R}$, $[0, 1)$, $[0, 1]$ are both strongly connected and essentially connected.

Non-strongly connected topologies

Theorem [Guthrie–Stone, 1973]

No Hausdorff connected space with a dispersion point has a maximal connected expansion.

- A dispersion point is the only cutpoint of a connected space.
- Every infinite Hausdorff maximal connected space has infinitely many cutpoints.

Observation

Strong connectedness is not preserved by connected subspaces since Knaster–Kuratowski fan / Cantor's leaky tent is a subspace of $\mathbb{R}^2$.

Submaximal, nodec, and $T_{\frac{1}{2}}$ spaces

Definition

Recall the following properties of a topological space X.

- X is *submaximal* if every its dense subset is open.
- X is *nodec* if every its nowhere dense subset is closed.
- X is $T_{\frac{1}{2}}$ if every its singleton is open or closed.

We have the following implications.

$$
\begin{tikzcd}
\text{maximal connected} \arrow[dr] & & \text{nodec} \\
& \text{submaximal} \arrow[ur] \arrow[dr] & \\
T_1 \text{ maximal} \arrow[ur] & & T_{\frac{1}{2}}
\end{tikzcd}
$$

Tree sums of topological spaces

Definition

Let $\langle X_i : i \in I \rangle$ be an indexed family of topological spaces, $\sim$ an equivalence on $\sum_{i \in I} X_i$, and $X := \sum_{i \in I} X_i/{\sim}$. We consider

- the canonical maps $e_i \colon X_i \to X$,
- the canonical quotient map $q \colon \sum_{i \in I} X_i \to X$,
- the *set of gluing points* $S_X := \{x \in X : |q^{-1}(x)| > 1\}$,
- the *gluing graph* G_X with vertices $I \sqcup S_X$ and edges of from $s \to_x i$ where $s \in S_X$, $i \in I$, and $x \in X_i$ such that $e_i(x) = s$.

We say that X is a *tree sum* if G_X is a tree, i.e. for every pair of distinct vertices there is a unique undirected path connecting them.

Example

A *wedge sum*, that is a space $\sum_{i \in I} X_i/{\sim}$ such that one point is chosen in each space X_i and $\sim$ is gluing these points together, is an example of a tree sum.

Tree sums of topological spaces

Proposition

A topological space X is naturally homeomorphic to a tree sum of a family of its subspaces $\langle X_i : i \in I \rangle$ if and only if the following conditions hold.

1. $\bigcup_{i \in I} X_i = X$,
2. X is inductively generated by embeddings $\{e_i \colon X_i \to X\}_{i \in I}$,
3. G is a tree, where G is the graph on $S \sqcup I$ satisfying
   - $S := \{x \in X : |\{i \in I : x \in X_i\}| \geq 2\}$,
   - $s \to i$ is an edge if and only if $s \in S$, $i \in I$, and $s \in X_i$.

Tree sums of topological spaces

Proposition

Let X be a tree sum of spaces $\langle X_i : i \in I \rangle$ such that every gluing point of X is closed. A subset $C \subseteq X$ is connected if and only if every $C \cap X_i$ is connected and G_C is connected (i.e. it is a subtree of G_X), where G_C is the subgraph of G_X induced by $I_C \sqcup S_C$,

$$I_C := \{i \in I : C \cap X_i \neq \emptyset\}, \qquad S_C := S_X \cap C.$$

In this case, C is the induced tree sum of spaces $\langle C \cap X_i : i \in i \rangle$.

Proposition

Let $\langle X, \tau \rangle := \sum_{i \in I} \langle X_i, \tau_i \rangle / \sim$ be a tree sum, $\mathcal{A} \subseteq \mathcal{P}(X)$. We put $\tau^* := \tau \vee \mathcal{A}$, $\tau_i^* := \tau_i \vee \{A \cap X_i : A \in \mathcal{A}\}$ for $i \in I$. If

- the set of gluing points S_X is closed discrete in $\langle X, \tau \rangle$,
- the family $\mathcal{A}$ is point-finite at every point of S_X,

then $\langle X, \tau^* \rangle = \sum_{i \in I} \langle X_i, \tau_i^* \rangle / \sim$, i.e. such expansion of a tree sum is a tree sum of the corresponding expansions.

Tree sums of maximal connected spaces

Theorem

Let X be a tree sum of spaces $\langle X_i : i \in I \rangle$ such that the set of gluing points is closed discrete.

1. If the spaces X_i are maximal connected, then X is such.
2. If the spaces X_i are strongly connected, then X is such.
3. If the spaces X_i are essentially connected, then X is such.

Examples

As a corollary we have that the spaces like $\mathbb{R}^\kappa$, $[0,1]^\kappa$, $\mathbb{S}^n$ are are strongly connected, and every topological tree graph is both strongly connected and essentially connected.

Finitely generated maximal connected spaces

Definition

A topological space X is called *finitely generated* or *Alexandrov* if every intersection of open sets is open. Equivalently, if

$$\overline{A} = \bigcup_{x \in A} \overline{\{x\}} \quad \text{for every } A \subseteq X.$$

- [Thomas, 1968] characterized finitely generated maximal connected spaces and introduced diagrams for visualizing them.
- [Kennedy–McCartan, 2001] reformulated the characterization in the language of so-called degenerate A-covers.
- We reformulate the characterization in the language of specialization preorder and graphs and also provide a visualization method.

Specialization preorder

Definition

The *specialization preorder* on a topological space X is defined by

$$x \leq y \;:\Longleftrightarrow\; \overline{\{x\}} \subseteq \overline{\{y\}}.$$

Facts

- Every open set is an upper set.
  Every closed set is a lower set.
- The converse holds if and only if X is finitely generated.
- The specialization preorder is an order if and only if X is T_0.
- Every isolated point is a maximal element,
  every closed point is a minimal element.

Finitely generated maximal connected spaces

Let X be a finitely generated $T_{\frac{1}{2}}$ space.

- The topology is uniquely determined by the specialization preorder, which is an order with at most two levels.
- Let us consider a graph G_X on X such that there is an edge between $x, y \in X$ if and only if $x < y$ or $y < x$.
- X is connected $\iff$ G_X is connected as a graph.
- X is maximal connected $\iff$ G_X is a tree.

Therefore, principal maximal connected spaces correspond to trees with fixed bipartition and also to tree sums of copies of the Sierpiński space.

Examples

The empty space, the one-point space, the Sierpiński space, principal ultrafilter spaces, principal ultraideal spaces.

I-spaces

Definition

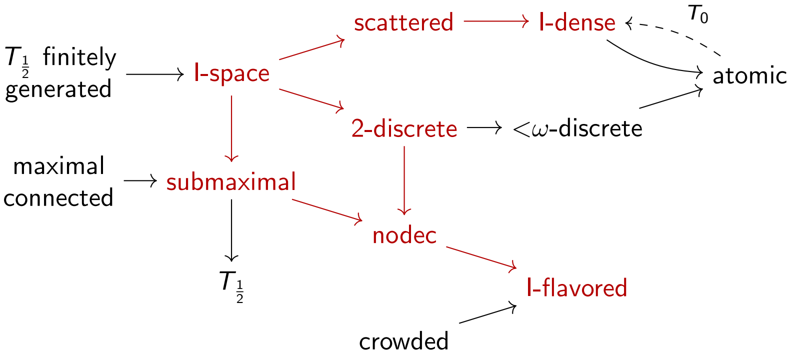
Let X be a topological space. By $I(X)$ we denote the set of all isolated points of X.

- X is an *I-space* if $X \setminus I(X)$ is discrete.
- X is *I-dense* if $\overline{I(X)} = X$.
- X is *I-flavored* if $\overline{I(X)} \setminus I(X)$ is discrete.

- I-spaces were considered in [Arhangel'skii–Collins, 1995].
- We are interested in *maximal connected I-spaces*, a class containing finitely generated maximal connected spaces.
- The term "maximal connected I-space" is unambiguous since I-spaces are closed under expansions.

l-spaces

- We have the following implications between the classes.
- The red part is a meet semilattice with respect to conjunction.



Maximal connected I-spaces

- The green part collapses in the realm of maximal connected spaces.

$$
\begin{tikzcd}
 & & \text{scattered} \arrow[r] & \text{I-dense} \arrow[dr, bend right] & \\
\begin{array}{c} T_{\frac{1}{2}} \text{ finitely} \\ \text{generated} \end{array} \arrow[r] & \text{I-space} \arrow[ur] \arrow[dr] \arrow[dd] & & & \text{atomic} \arrow[ul, dashed, bend right, "T_0"'] \\
 & & \text{2-discrete} \arrow[r] \arrow[dd] & {<}\omega\text{-discrete} \arrow[ur] & \\
\begin{array}{c} \text{maximal} \\ \text{connected} \end{array} \arrow[r] & \text{submaximal} \arrow[dr] \arrow[dd] & & & \\
 & & \text{nodec} \arrow[dr] & & \\
 & T_{\frac{1}{2}} & & \text{I-flavored} & \\
 & & \text{crowded} \arrow[ur] & &
\end{tikzcd}
$$

I-extensions

Definition

Let X be a topological space, Y a set disjoint with X, and $\mathcal{F} := \langle \mathcal{F}_y : y \in Y \rangle$ an indexed family of filters on $I(X)$. Let $\widehat{X}$ be the space with universe $X \cup Y$ and the following topology:

$$A \subseteq \widehat{X} \text{ is open } \iff \begin{cases} A \cap X \text{ is open in } X, \\ A \cap I(X) \in \mathcal{F}_y \text{ for every } y \in A \cap Y. \end{cases}$$

The space $\widehat{X}$ is called the *I-extension* of X by $\mathcal{F}$.

Observations

- X becomes an open subspace of $\widehat{X}$.
- I-spaces are precisely I-extensions of discrete spaces in a canonical way.

Maximal connected I-extensions

Proposition

An I-extension of a maximal connected space is maximal connected if and only if it is an I-extension by a family of ultrafilters.

Proposition

Let X be an I-space. If $A \subseteq X$, then $\overline{A}$ is an I-extension of A.

Corollary

Let X be a maximal connected I-space. If $A \subseteq X$ is connected, then $\overline{A}$ is an I-extension of A by a family of ultrafilters.

Towards characterization of maximal connected I-spaces

Definition

Let X be a topological space. We define the following, so that for every ordinal α we have

- $\mathcal{D}_\alpha$ is a decomposition of X into connected subsets,
- E_α is the corresponding equivalence,
- G_α is a graph on $\mathcal{D}_\alpha$ with $\langle D, x\rangle$ being an edge $D \to D'$ for $D \neq D' \in \mathcal{D}_\alpha$ if and only if $\overline{D} \cap D' \ni x$,
- $\mathcal{D}_0 := \{\{x\} : x \in X\}$,
- $\mathcal{D}_{\alpha+1} := \{\bigcup \mathcal{C} : \mathcal{C} \text{ is an undirected component of } G_\alpha\}$,
- $E_\alpha := \bigcup_{\beta<\alpha} E_\alpha$ for limit α.

We denote the smallest α such that $\mathcal{D}_\alpha = \mathcal{D}_{\alpha+1}$ by $\rho(X)$.

Towards characterization of maximal connected I-spaces

Theorem

Let X be a maximal connected I-space, let α be an ordinal. Let $D \in \mathcal{D}_{\alpha+1}$ and let $\mathcal{C}$ be the component of G_α such that $D = \bigcup \mathcal{C}$.

1. $\overline{C}$ is an I-extension of C by a family of ultrafilters for every $C \in \mathcal{C}$. If $\alpha = 0$, then the ultrafilters are principal. If $\alpha > 0$, then the ultrafilters are free.
2. The graph $G_\alpha \restriction \mathcal{C}$ is a tree.
3. D is the tree sum of its subspaces $\{\overline{C} : C \in \mathcal{C}\}$. The set of gluing points is closed discrete.

Therefore, the members of $\mathcal{D}_{\rho(X)}$ are obtained by iteratively forming tree sums of ultrafilter I-extensions.

Intersections of connected subsets

In the proof of the previous theorem, the following properties of maximal connected spaces are needed.

Theorem [Neumann-Lara, Wilson; 1986]

Let X be an essentially connected space. If $A, B \subseteq X$ are connected, then $A \cap B$ is connected as well.

Corollary

Let X be an maximal connected space. If $A, B \subseteq X$ are disjoint and connected, then $|A \cap B| \leq 1$.

Proof.
We have $\overline{A} \cap \overline{B} \subseteq (\overline{A} \setminus A) \cup (\overline{B} \setminus B)$, which is a closed discrete set since X is submaximal.

Towards characterization of maximal connected I-spaces

Proposition

Every maximal connected space having only finitely many nonisolated points is an I-space satisfying $|\mathcal{D}_1| < \omega$ and $|\mathcal{D}_2| \leq 1$. Therefore, it is a finite tree sum of free ultrafilter I-extensions of finitely generated maximal connected spaces.

Because of the previous results, a maximal connected I-space X such that $|\mathcal{D}_{\rho(X)}| \leq 1$ may be called *inductive*. We shall conclude with an example of a non-inductive maximal connected I-space.

Towards characterization of maximal connected I-spaces

Example

Let $f\colon X \to Y$ be a bijection between two disjoint sets, let $\mathcal{U}$ be a free ultrafilter on X. Let $\widehat{X}$ be the I-extension of X with discrete topology by the family $\langle \mathcal{F}_y : y \in Y \rangle$ where

$$\mathcal{F}_y := \{U \in \mathcal{U} : f^{-1}(y) \in U\} \text{ for every } y \in Y$$

The space $\widehat{X}$ is an example of a non-inductive maximal connected I-space.

References I

- Arhangel'skii A. V., Collins P. J., *On submaximal spaces*. Topology Appl. 64 (1995), 219–241.
- Cameron D. E., *Maximal and minimal topologies*, Trans. Amer. Math. Soc. 160 (1971), 229–248.
- Guthrie J. A., Reynolds D. F., Stone H. E., *Connected expansions of topologies*, Bull. Austral. Math. Soc. 9 (1973), 259–265.
- Guthrie J. A., Stone H. E., *Spaces whose connected expansions preserve connected subsets*, Fund. Math. 80 (1973), 91–100.
- Guthrie J. A., Stone H. E., Wage M. L., *Maximal connected expansions of the reals*, Proc. Amer. Math. Soc. 69 (1978), 159–165.
- Hildebrand S. K., *A connected topology for the unit interval*. Fund. Math. 61 (1967), 133–140.
- Kennedy G. J., McCartan, S. D., *Maximal connected principal topologies*, Math. Proc. R. Ir. Acad. 101A (2001), no. 2, 163–166.

References II

- Neumann-Lara V., Wilson R. G., *Some properties of essentially connected and maximally connected spaces*. Houston J. Math. 12 (1986), 419–429.
- Simon P., *An example of maximal connected Hausdorff space*, Fund. Math. 100 (1978), no. 2, 157–163.
- Thomas J. P., *Maximal connected topologies*, J. Austral. Math. Soc. 8 (1968), 700–705.

Thank you for your attention.