Homology of generalized generalized configuration spaces

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joint work with R. Sazdanović (NCSU) and V. de Silva (Pomona)

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the chromatic polynomial

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Definition

The chromatic function $P_G(\lambda)$ of the graph $G = (V, E)$ is the number of ways, given λ colors, to color each vertex $v \in V$ so that adjacent vertices have distinct colors.

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Originally introduced by Birkhoff 1912

to prove the Four Color Theorem.
(Birkhoff-Lewis proved that 5 colors suffice.)

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Elementary fact

$P_G(\lambda)$ is polynomial in λ, of degree $|V|$.

the chromatic polynomial

Theorem (deletion-contraction formula)

Given a simple graph $G = (V, E)$ *and* $e \in E$*, let*

$$G - e = (V, E \setminus \{e\})$$

and $G_{/e}$ *be the graph given by contracting e to a point. Then*

$$P_G = P_{G-e} - P_{G_{/e}}$$

the chromatic polynomial

P_G detects combinatorial and topological features

- $|V|$
- $|E|$

P_G does not detect

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- isomorphism type

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- homotopy type

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- ...

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- isomorphism type
- homotopy type

A more sophisticated invariant is called for. . .

numerical invariants, characteristic polynomial, Tutte polynomial, . . .

graph configuration spaces and graph homology

configuration spaces

Definition (Fadell, Neuwrith)

Configuration space on a topological space X is the space of n-tuples of distinct points on X

$$Conf(X,n) = \left\{ (x^1, x^2, \ldots, x^n) \in X^n \,\middle|\, x^i \neq x^j \text{ if } i \neq j \right\}$$

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- X manifold $\Rightarrow$ $Conf(X, n)$ manifold

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- applications to harmonic analysis, complex analysis, geometry, physics
- X manifold $\Rightarrow Conf(X,n)$ manifold
- natural action of S_n on $Conf(X,n)$

graph configuration space

Conf(X, n)

Start with X^n, remove all diagonals $\Delta_{ij} = \{x^i = x^j\}$ for $i \neq j$.

graph configuration space

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graph configuration space (Eastwood-Huggett)

- $G = (V, E)$ graph, X topological space, $n = |V|$
- For each $e = [v_i, v_j] \in E$, set

$$\Delta_e := \left\{ (x^1, \ldots, x^n) \middle| x^i = x^j \right\} \subset X^n$$

- $M_G(X) := X^n \setminus \bigcup_{e \in E} \Delta_e$
- $M_{K_n}(X) = Conf(X, n)$; $M_{\underbrace{\cdot\ \cdot\ \cdot\ \cdot\ \cdot}_{n}}(X) = X^n$

graph configuration space

Theorem (Eastwood-Huggett)

The Euler characteristic $\chi(M_G(X))$ satisfies:

$$\chi(M_G(X)) = \chi(M_{G-e}(X)) - \chi(M_{G/_e}(X))$$

Corollary

$\chi(M_G(X)) = P_G(\chi(X))$.

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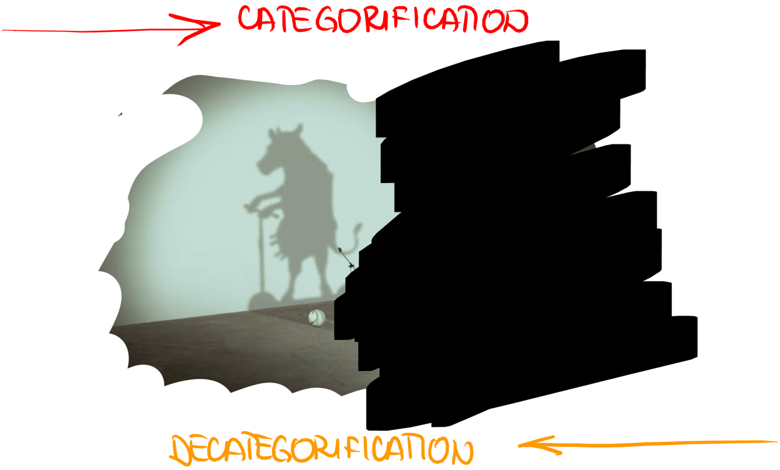
The homology $H_*(M_G(X))$ is a categorification of the value $P_G(\chi(X))$.

categorification (with apologies to Plato)



categorification (with apologies to Plato)

CATEGORIFICATION

DECATEGORIFICATION

categorification

the (fuzzy, vague) idea

Given a structure S, assign a category $C(S)$, a categorification $H : S \to Obj(C(S))$, a characteristic $\chi : Obj(C(S)) \to S$ so that

$$\chi(H(s)) = s$$

categorification

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Try to find richer structure in $C(S)$ than we saw in S.

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for example

- Direct sum of vector spaces categorifies addition of positive integers, with $H : m \to V^m$, $\chi = \dim$
- Short exact sequence of abelian groups categorifies subtraction of positive integers, with $\chi = \text{rank}$
- Singular homology $H_*(Y)$ categorifies (topological) Euler number of Y, with $\chi = \sum(-1)^i \,\text{rank}\, H_i$

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main difficulty to obtain useful categorifications

Coming up with differentials!

generalizing to simplicial complexes

simplicial configuration space

- S simplicial complex with n vertices $v_1, \ldots, v_n$
- For each simplex $\sigma^k = [v_{i_0} \cdots v_{i_k}]$, set

$$\Delta_\sigma := \left\{ (x^1, \ldots, x^n) \middle| x^{i_0} = \cdots = x^{i_k} \right\}$$

- The simplicial configuration space is

$$M(S, X) := X^n \setminus \bigcup_{\sigma \notin S} \Delta_\sigma$$

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- $M(\underbrace{\cdot \cdots \cdot}_{n}, X) = Conf(X, n)$; $M(\Delta^p, X) = X^{p+1}$
- $M_G(X) = M(I(G), X)$ where $I(G)$ is the independence complex of G

deletion-contraction sequence

Definition

Given a simplicial complex S and $\sigma \in S$, define

$S_{/\sigma}$ *the simplicial complex obtained by contracting σ.*

$\mathrm{St}(\sigma)$ *the collection of all simplices with σ as a face.*

$S - \sigma$ *the simplicial complex $S \setminus \mathrm{St}(\sigma)$.*

Theorem (C-S-dS)

Let S be a simplicial complex, X a manifold of dimension m, and $\sigma^k \in S$. There is a long exact sequence in homology

$$\cdots \to H_p(M(S-\sigma, X)) \to H_p(M(S, X)) \to H_{p-mk}(M(S_{/\sigma} - \mathrm{St}(v), X)) \to \cdots$$

where v is the vertex to which σ has been identified.

deletion-contraction formula(e)

Theorem (C-S-dS: deletion-contraction formula)

Let S be a simplicial complex, X a manifold of dimension m, and $\sigma^k \in S$. Then

$$\chi(M(S,X)) = \chi(M(S-\sigma,X)) + (-1)^{mk}\chi(M(S_{/\sigma} - \mathrm{St}(v),X))$$

Theorem (C-S-dS: addition-contraction formula)

Let S be a simplicial complex, X a manifold of dimension m, and $\sigma^k \notin S$ a simplex all of whose faces are in S. Then

$$\chi(M(S,X)) = \chi(M(S\cup\sigma,X)) - (-1)^{mk}\chi(M(S_{/\sigma} - \mathrm{St}(v),X))$$

deletion-contraction formula

Corollary

$\chi(M(S,X))$ is polynomial in $\chi(X)$; in fact

$$\chi(M(S,X)) = (-1)^{mn}\chi(X)^n + a_{n-1}\chi(X)^{n-1} + \cdots + a_1\chi(X) + 0$$

deletion-contraction formula

idea of proof

The Leray long exact sequence:
B manifold, A closed submanifold, codim $A = m$,

$$\cdots \to H_k(B \setminus A) \to H_k(B) \to H_{k-m}(A) \to H_{k-1}(B \setminus A) \cdots$$

In cohomology with compact supports,

$$\cdots \to H_c^k(B \setminus A) \to H_c^k(B) \to H_c^k(A) \to H_c^{k+1}(B \setminus A) \to \cdots$$

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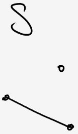
Use

$$B = M(S, X) \qquad B \setminus A = M(S, X) \setminus \Delta_\sigma$$

A is homeomorphic to $M(S_{/\sigma} - \mathrm{St}(v), X)$

computation



$$\chi_c(M(S,X)) = \chi_c(X)^3 - 2\chi_c(X)^2 + \chi_c(X)$$

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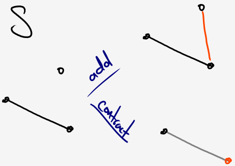

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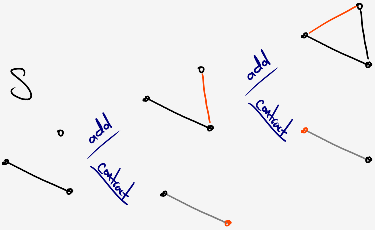

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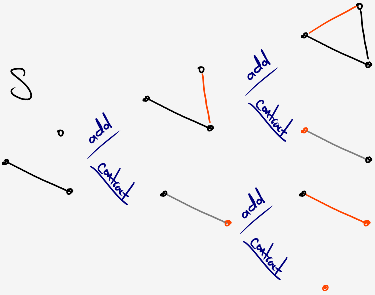

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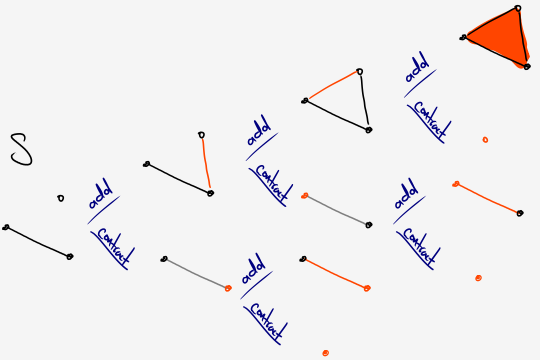

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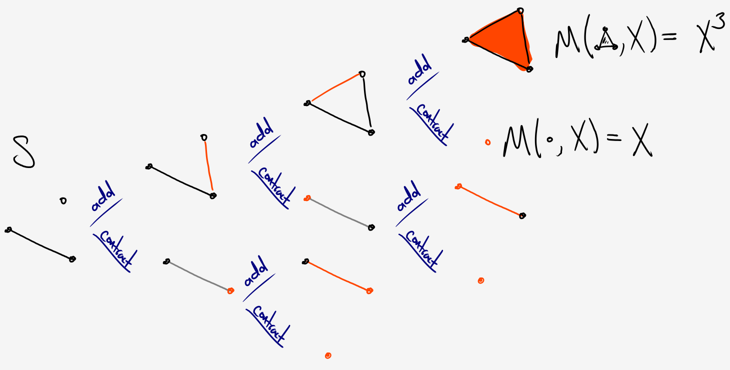

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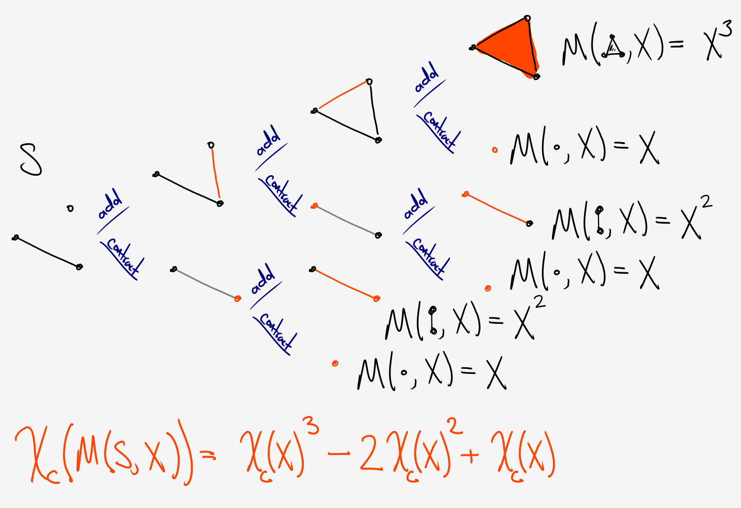

state-sum formula

Definition

A state on S is a set T of simplices on $\{v_1, \ldots, v_n\}$ not in S.

Given a state T, set $k(T)$ to be the number of connected components of $\{v_1, \ldots, v_n\} \cup (\cup T)$.

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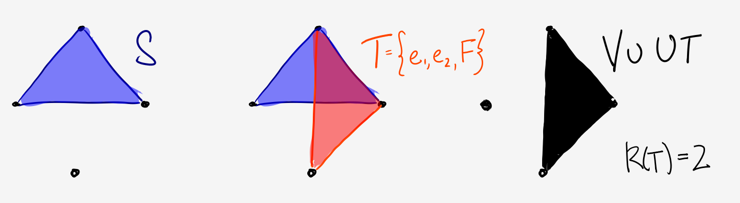

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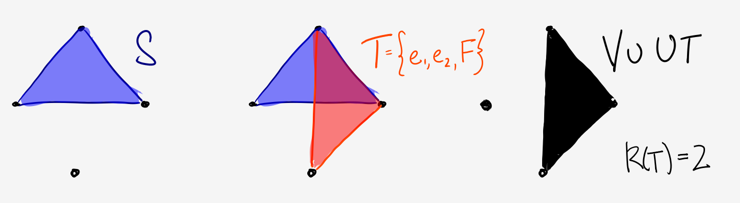

Theorem

$$\chi(M(S,X)) = (-1)^{mn} \sum_T (-1)^{|T|+m\cdot k(T)} \chi(X)^{k(T)}$$

(Compare to chromatic polynomial: $P_G(\lambda) = \sum_{s\subseteq E} (-1)^{|s|} \lambda^{[G:s]}$)

categorification

structure of $H(M(S, X))$

functoriality in X

- H_* is functorial (covariant); H_c^* is functorial (contravariant)
- $f : X \to Y$ induces $f^n : X^n \to Y^n$
- f injective $\Rightarrow f^n : M(S, X) \to M(S, Y)$ (also injective)
- induced maps on homology and cohomology:
$$(f^n)_* : H_*(M(S, X)) \to H_*(M(S, Y))$$
$$(f^n)^* : H_c^*(M(S, Y)) \to H_c^*(M(S, X))$$
- obtain functors $H_*(M(S, \cdot)), H_c^*(M(S, \cdot))$

$$\text{(manifolds, cont. inj.)} \to \text{(graded ab. gps., degree 0 hom.)}$$

functoriality in S

- $S_1 \subset S_2$ induces inclusions $i : M(S_1, X) \to M(S_2, X)$
and projection $\pi : M(S_2, X) \to M(S_1, X)$
- Hence $\pi^* : H_c^*(M(S_1, X)) \to H_c^*(M(S_2, X))$ is injective.

future directions

- Develop computational tools.
- Interpret $\chi(M(S, X))$ as "colorings of S (S^c?) with $\chi(X)$ colors".
- Which topological properties are detected by the polynomial?
- Is the homology theory richer than the polynomial?
- Functoriality
  1. Simplicial maps?
  2. Exploit functoriality to obtain polynomial or numerical invariants.
- Relations to cell-complex invariants (Bott, Tutte-Krushkal-Renardy)
- Is there a purely algebraic construction of the homology theory?

THANKS!