Combinatorics of spoke systems for Fréchet-Urysohn points

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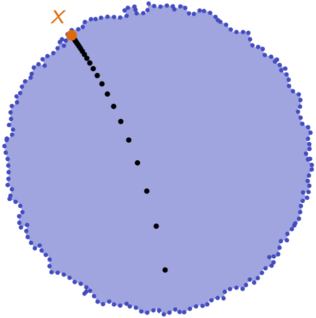
What are Fréchet-Urysohn points?

Definition

X is *Fréchet-Urysohn* at x if whenever $A \subseteq X$ and $x \in \overline{A}$, there exists a sequence (x_n) in A that converges to x.



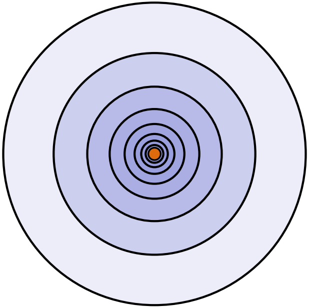
Fréchet-Urysohn point

$(x_n) \to x$

Some examples

Definition

X is *first-countable* at x if there exists a countable neighbourhood base for x. Equivalently, there exists a descending neighbourhood base (B_n) for x.

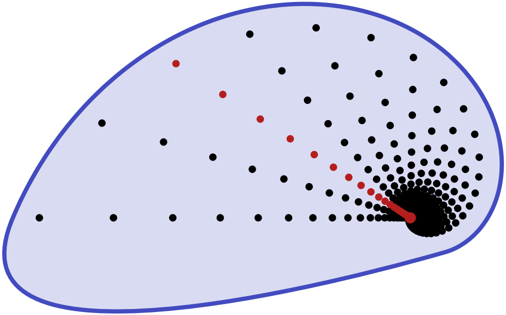


First countable point

More examples

Definition

The *sequential hedgehog* is the space obtained by quotienting the limit points of a countable sum of convergent sequences.



More examples

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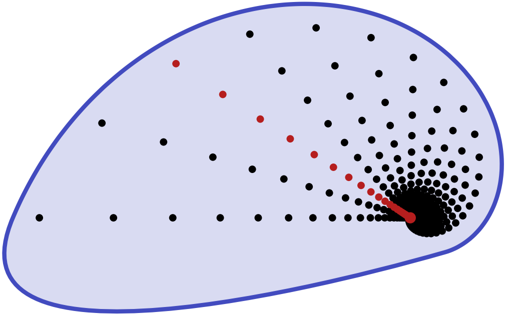

Proposition

The sequential hedgehog is Fréchet-Urysohn but not first-countable.

Spokes

Definition

A *spoke* of a point x in a space X is a subspace $S \subseteq X$ where $N_x := \bigcap \mathcal{N}_x \subseteq S$ and x is first-countable with respect to S.

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Lemma

Let (x_n) be a sequence in $X \setminus N_x$ that converges to x. Then $\mathbb{S}_{(x_n)} := N_x \cup \{x_n : n \in \mathbb{N}\}$ is a spoke for x.

Spokes

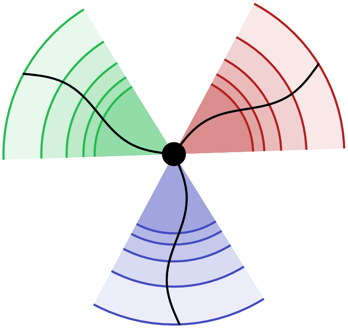
Definition

A *spoke system of x* is a collection $\mathfrak{S}$ of spokes of x such that

$$\left\{ \bigcup_{S \in \mathfrak{S}} U_S : \forall S \in \mathfrak{S}, U_S \in \mathcal{N}_x^S \right\}$$

is a neighbourhood base of x with respect to X.

Spokes



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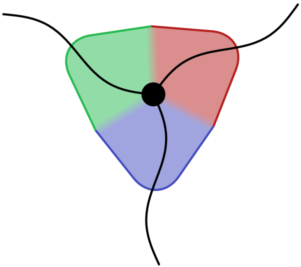

Basic neighbourhood

Spokes

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Proposition

A collection $\mathfrak{S}$ of spokes of x is a spoke system if and only if for every $A \subseteq X$ with $x \in \overline{A}$, there exists an $S \in \mathfrak{S}$ such that $x \in \overline{A \cap S}$.

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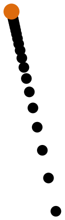
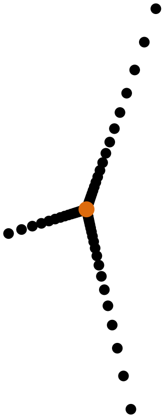
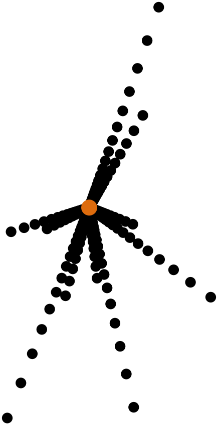
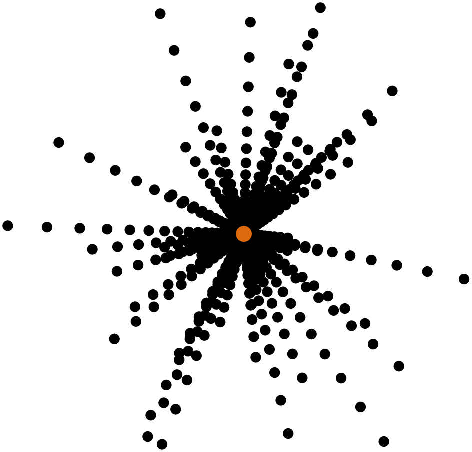
Corollary

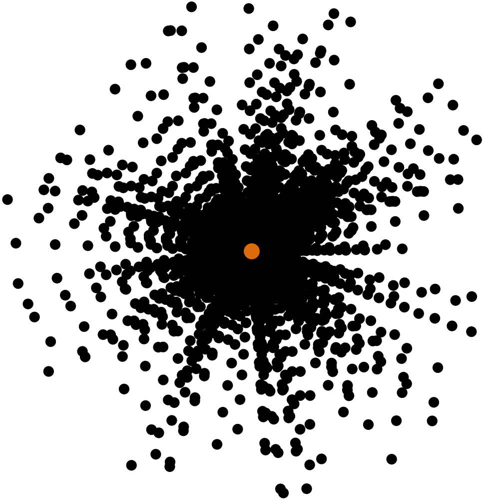
Every point with a spoke system is Fréchet-Urysohn.

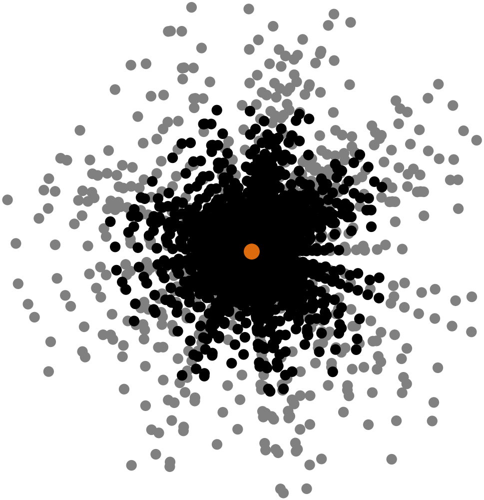












Constructing spokes

Theorem

x is Fréchet-Urysohn if and only if x has a spoke system $\mathfrak{S}$ such that $x \notin \overline{(S \cap T) \setminus N_x}$ for all distinct $S, T \in \mathfrak{S}$.

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Proof.

If X is Fréchet-Urysohn at x and not quasi-isolated (i.e. N_x is open), define

$$\mathcal{T} := \{f : \mathbb{N} \to X \setminus N_x \mid f \text{ is injective}\}$$
$$\mathcal{A} := \{\mathcal{F} \subseteq \mathcal{T} : \forall f, g \in \mathcal{F} \text{ distinct}, ran(f) \cap ran(g) \text{ is finite}\}$$

By Zorn's lemma, pick a maximal $\mathcal{F} \in \mathcal{A}$ and define for all $f \in \mathcal{F}, \mathbb{S}_f := N_x \cup \mathrm{ran}(f)$. Then by maximality, $\mathfrak{S} := \{\mathbb{S}_f : f \in \mathcal{F}\}$ is a spoke system for x. Moreover, for all $f, g \in \mathcal{F}$ distinct, $x \notin \overline{(\mathbb{S}_f \cap \mathbb{S}_g) \setminus N_x}$ since $\mathcal{F} \in \mathcal{A}$. □

(Almost-)independence

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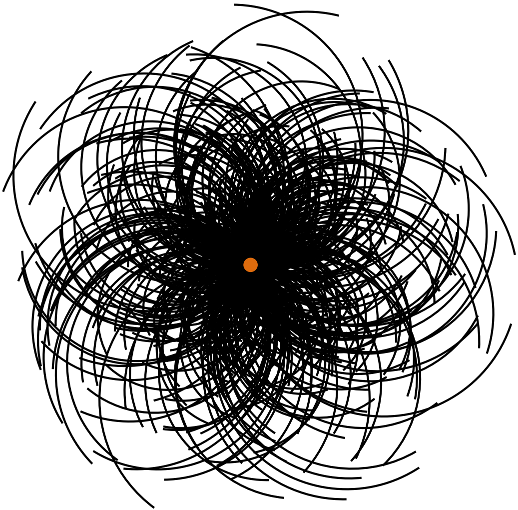

Summary of spoke systems

A spoke system $\mathfrak{S}$ of $x \in X$:

- consists of first-countable (i.e. *nice*) approximations;
- generates a neighbourhood base in the original space, via:

$$\left\{ \bigcup_{S \in \mathfrak{S}} U_S : \forall S \in \mathfrak{S}, U_S \in \mathcal{N}_x^S \right\}$$

- gives witnesses for sequences: if $x \in \overline{A}$ then $x \in \overline{A \cap S}$ for some $S \in \mathfrak{S}$, and we can now easily find a convergent sequence in $A \cap S$.

Summary of spoke systems

The language of this framework consists of our spokes in $\mathfrak{S}$, arbitrary subsets $A \subseteq X$ and how they intersect. We introduce some notation.

Definition

Given subsets $A, B \subseteq X$ and a point $x \in X$, we write:

- $A \perp_x B$ if $A \cap B = N_x$.
- $A \#_x B$ if $x \in \overline{(A \cap B) \setminus N_x}$.

We omit the x when there is no ambiguity.

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From now on, we will assume that our spoke systems are:

- Almost-independent: $S \not\# T$ for all distinct $S, T \in \mathfrak{S}$.
- Non-trivial: $X \# S$ for all $S \in \mathfrak{S}$.

Stronger convergence properties

Definition (α_4 / strongly Fréchet)

A point x is α_4 if whenever (σ_n) is a sequence of (disjoint) sequences in $X \setminus N_x$ that converges to x, then there exists another sequence $\sigma \to x$ such that $\operatorname{ran}(\sigma_n) \cap \operatorname{ran}(\sigma) \neq \emptyset$ for infinitely-many n.

If x is α_4 and Fréchet-Urysohn, we say it is strongly Fréchet.

Definition (α_2)

A point x is α_2 if whenever (σ_N) is a sequence of (disjoint) sequences in $X \setminus N_x$ that converges to x, then there exists another sequence $\sigma \to x$ such that $\operatorname{ran}(\sigma_n) \cap \operatorname{ran}(\sigma)$ is infinite, for all $n \in \omega$.

Spoke system characterisations

Theorem

If x is Fréchet-Urysohn, the following are equivalent:

- *x is α_4.*
- *For any spoke system $\mathfrak{S}$ and any countably-infinite $\mathcal{S} \subseteq \mathfrak{S}$, there exists a $T \in \mathfrak{S}$ such that $T \not\perp S$ for infinitely-many $S \in \mathcal{S}$.*

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Theorem

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- *x is α_2.*
- *For any spoke system $\mathfrak{S}$ and countably-infinite $\mathcal{S} \subseteq \mathfrak{S}$, there exists an $A \subseteq X$ such that:*
  1. *$A \# S$ for all $S \in \mathcal{S}$, and*
  2. *for all $B \subseteq A$, if $B \not\perp S$ for infinitely-many $S \in \mathcal{S}$, then $B \# T$ for some $T \in \mathfrak{S}$.*

Unbounded families from strongly-Fréchet points

Recall that an unbounded family is a family $\mathcal{B} \subseteq {}^{\omega}\omega$ that is unbounded with respect to the quasi-order $\leq^*$.

Theorem

Let x be a strongly-Fréchet, non-first-countable point in a space X and let $\mathfrak{S}$ be a spoke system of x and let (S_n) be an injective sequence in $\mathfrak{S}$. For each $n \in \omega$, pick a descending neighbourhood base $(U_{n,k})_{k\in\omega}$ of x with respect to S_n. Define for each $T \in \mathfrak{S} \setminus \{S_n : n \in \omega\}$:

$$f_T : \omega \to \omega, n \mapsto \sup(k \in \omega : U_{n,k} \cap T \neq N_x)$$

Then $\{f_T : T \in \mathfrak{S} \setminus \{S_n : n \in \omega\}\}$ is unbounded.

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Then $\{f_T : T \in \mathfrak{S} \setminus \{S_n : n \in \omega\}\}$ is unbounded.

Corollary

If x is a strongly-Fréchet, non-first-countable point, then every spoke system of x has cardinality at least $\mathfrak{b}$.

Unbounded families from strongly-Fréchet points

Theorem

If x is a Fréchet-Urysohn, α_2-point, then the unbounded family $\mathcal{B}$ obtained from the previous theorem is hereditarily-unbounded*: for every infinite $A \subseteq \omega$, the family $\{f|_A : f \in \mathcal{B}\}$ is unbounded in $({}^A\omega, \leq^*)$.*