Laminations of the Unit Disk and Cubic Julia Sets

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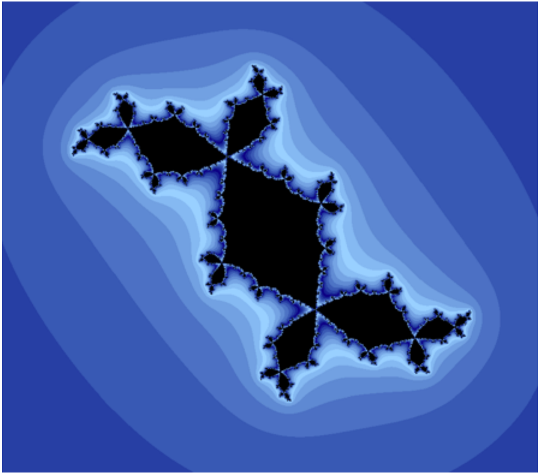
The Douady Rabbit $z \mapsto z^2 - 0.12 + 0.78i$



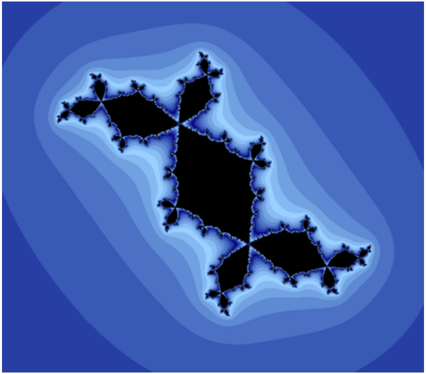
Julia sets by FractalStream

The Rabbit Lamination

Hyperbolic lamination pictures courtesy of Clinton Curry and Logan Hoehn

Rabbit Juilia Set and Rabbit Lamination

Family resemblance?



Outline

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Julia and Fatou Sets of Polynomials

Definitions:

- Basin of attraction of infinity: $B_\infty := \{z \in \mathbb{C} \mid P^n(z) \to \infty\}$.
- Filled Julia set: $K(P) := \mathbb{C} \setminus B_\infty$.
- Julia set: $J(P) :=$ boundary of B_∞ = boundary of $K(P)$.
- Fatou set: $F(P) := \mathbb{C}_\infty \setminus J(P)$.

Theorems (Facts):

- $J(P)$ is nonempty, compact, and perfect.
- $K(P)$ is full (does not separate $\mathbb{C}$).
- Attracting orbits are in Fatou set.
- Repelling orbits are in Julia set.

Examples: $P(z) = z^2$; $P(z) = z^d$, $d > 2$; $P(z) = z^2 - 1$, etc.
Assume: $J(P)$ is connected.

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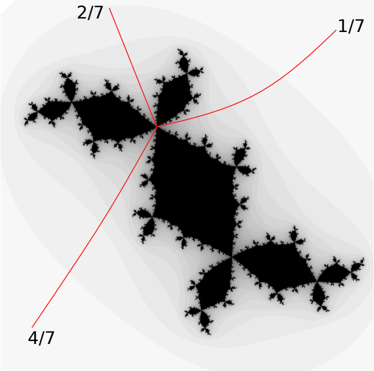
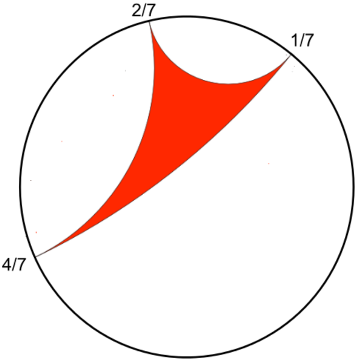
The Rabbit Juilia Set and Rabbit Triangle

External Rays

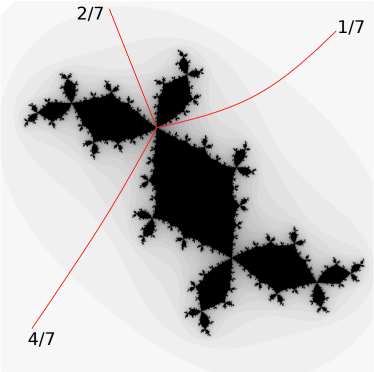


Landing Angles



The Rabbit Juilia Set and Rabbit Lamination

Down the rabbit hole!



Böttkher's Theorem

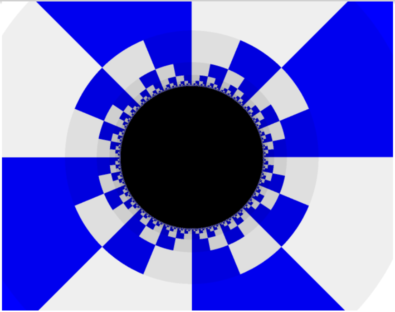
By $\mathbb{D}_\infty$, "the disk at infinity," we mean $\mathbb{C}_\infty \setminus \overline{\mathbb{D}}$, the complement of the closed unit disk.

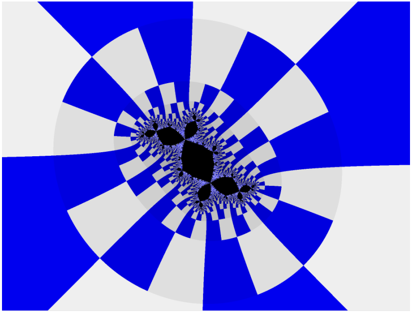
Theorem (Böttcher)

Let P be a polynomial of degree d. If the filled Julia set K is connected, then there is a conformal isomorphism

$$\phi : \mathbb{D}_\infty \to B_\infty,$$

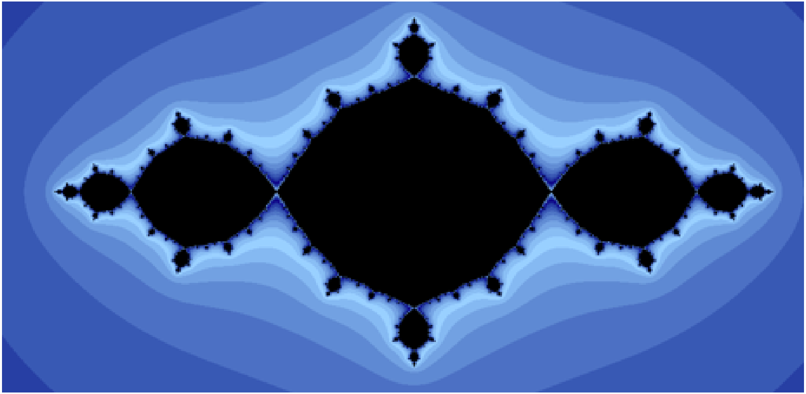
tangent to the identity at ∞, that conjugates P to $z \to z^d$.



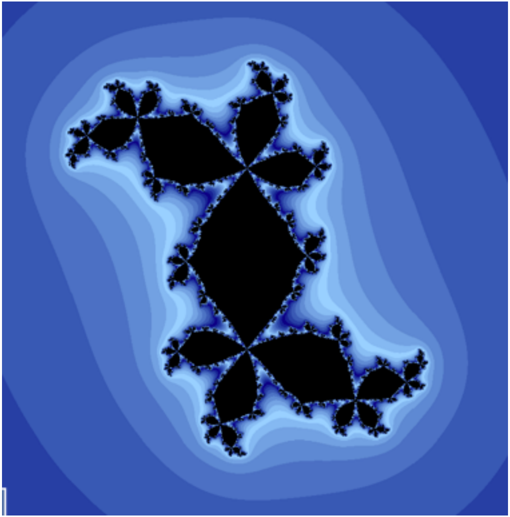


$$\begin{array}{ccc} \mathbb{D}_\infty & \xrightarrow{z \mapsto z^d} & \mathbb{D}_\infty \\ \downarrow \phi & & \downarrow \phi \\ B_\infty & \xrightarrow[P]{} & B_\infty \end{array}$$

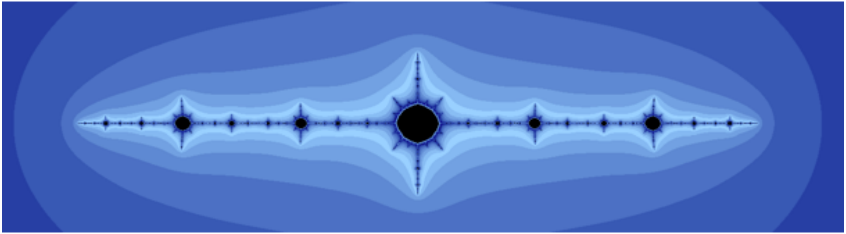
Basillica $z \mapsto z^2 - 1$



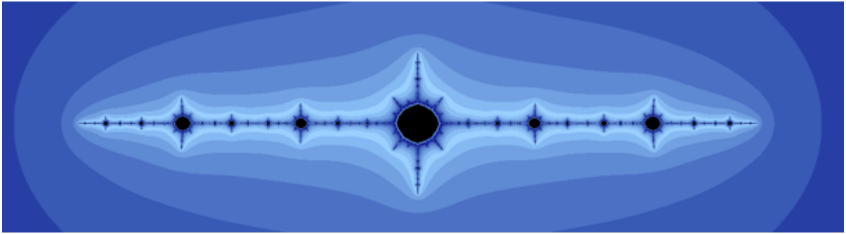
Dragon $z \mapsto z^2 - 0.28136 + 0.5326i$



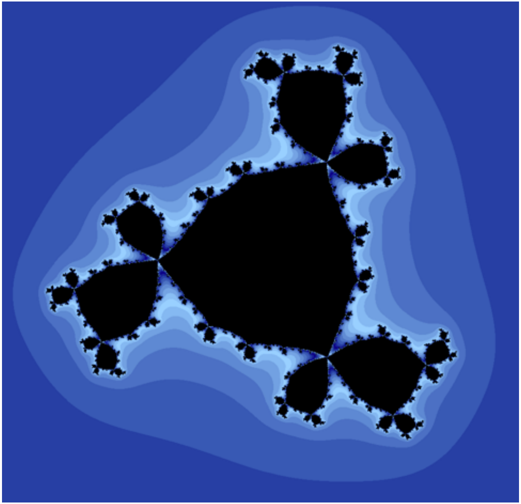
Airplane $z \mapsto z^2 - 1.75$



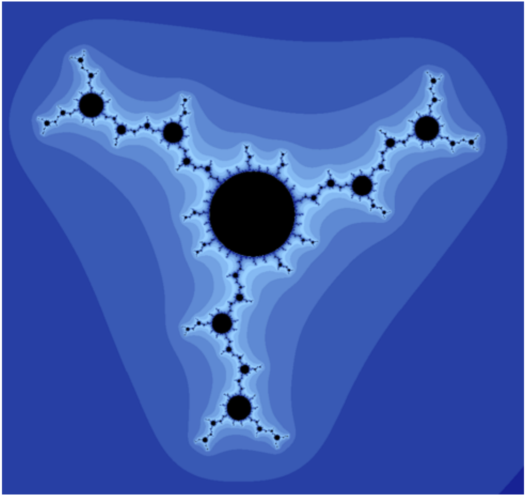
Airplane and B-17 Yankee Lady 1



Cubic Rabbit $z \mapsto z^3 + 0.545 + 0.539i$

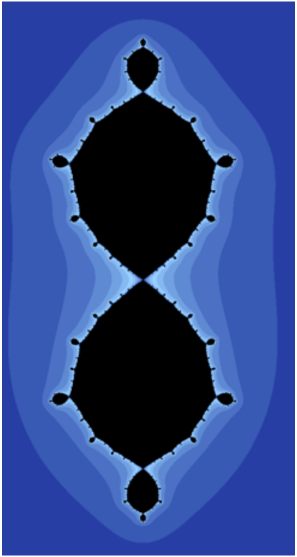


Helicopter $z \mapsto z^3 - 0.2634 - 1.2594i$



Cubic Bug

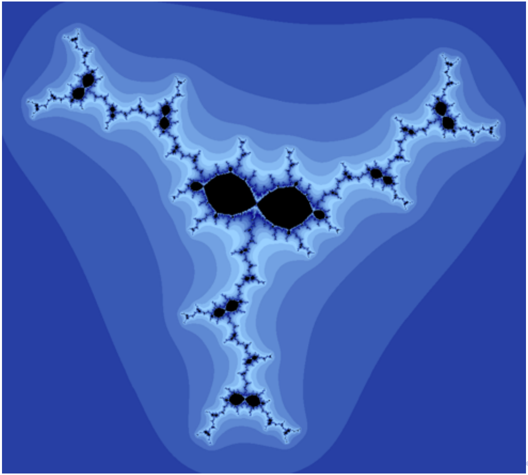
$z \mapsto z^3 + \frac{\sqrt{2}}{2} i\, z^2$



Cubic Simple Type 1 IRT

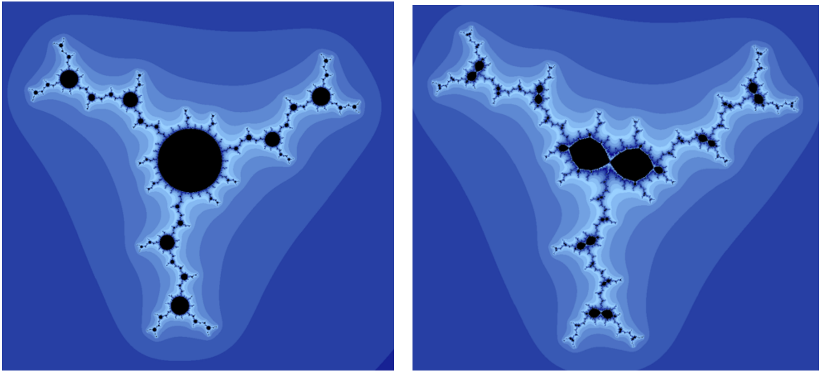
$z \mapsto z^3 + 3fz^2 + g$

$f = -0.167026 + 0.0384441i$ and $g = -0.0916222 - 1.2734i$



Comparison

$z \mapsto z^3 + c$ $\qquad$ $z \mapsto z^3 + 3fz^2 + g$



$f = -0.167026 + 0.0384441i$ and $g = -0.0916222 - 1.2734i$
$c = -0.2634 - 1.2594i$

Laminations of the Disk

- Laminations were introduced by William Thurston as a way of encoding connected polynomial Julia sets.

Definition

- A *lamination* $\mathcal{L}$ is a collections of chords of $\overline{\mathbb{D}}$, which we call *leaves*, with the property that any two leaves meet, if at all, in a point of $\partial\mathbb{D}$, and
- such that $\mathcal{L}$ has the property that

$$\mathcal{L}^* := \partial\mathbb{D} \cup \{\cup\mathcal{L}\}$$

is a closed subset of $\overline{\mathbb{D}}$.
- We allow *degenerate* leaves – points of $\partial\mathbb{D}$.

?Lamination to Julia Set?

The Beginning: Dynamics on the Circle

- Consider special case $P(z) = z^d$ on the unit circle $\partial\mathbb{D}$.
- $z = re^{2\pi t} \mapsto r^d e^{2\pi(dt)}$.
- Angle $2\pi t \mapsto 2\pi(dt)$.
- Measure angles in revolutions: then $t \mapsto dt \pmod 1$ on $\partial\mathbb{D}$.
- Points on $\partial\mathbb{D}$ are coordinatized by $[0, 1)$.

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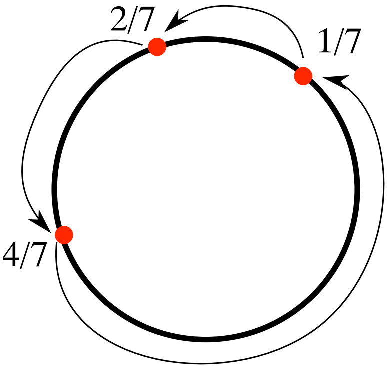
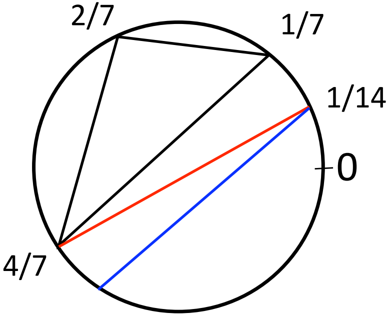
σ_d Dynamics on the Circle

- $\sigma_2 : t \mapsto 2t \pmod 1$, angle-doubling.



Induced map σ_d on Laminations

- If $\ell \in \mathcal{L}$ is a leaf, we write $\ell = \overline{ab}$, where a and b are the endpoints of ℓ in $\partial\mathbb{D}$.
- We let $\sigma_d(\ell)$ be the chord $\overline{\sigma_d(a)\sigma_d(b)}$.
- If it happens that $\sigma_d(a) = \sigma_d(b)$, then $\sigma_d(\ell)$ is a point, called a *critical value* of $\mathcal{L}$, and we say ℓ is a *critical* leaf.



Sibling Invariant Laminations

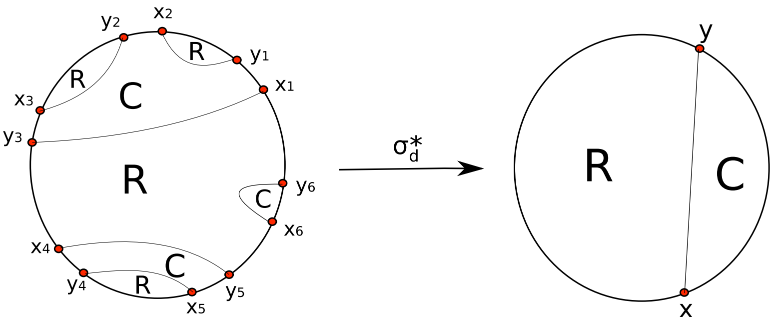
Definition (Sibling Invariant Lamination)

A lamination $\mathcal{L}$ is said to be *sibling d-invariant* (or simply *invariant* if no confusion will result) provided that

1. (Forward Invariant) For every $\ell \in \mathcal{L}$, $\sigma_d(\ell) \in \mathcal{L}$.
2. (Backward Invariant) For every non-degenerate $\ell' \in \mathcal{L}$, there is a leaf $\ell \in \mathcal{L}$ such that $\sigma_d(\ell) = \ell'$.
3. (Sibling Invariant) For every $\ell_1 \in \mathcal{L}$ with $\sigma_d(\ell_1) = \ell'$, a non-degenerate leaf, there is a full sibling collection $\{\ell_1, \ell_2, \ldots, \ell_d\} \subset \mathcal{L}$ such that $\sigma_d(\ell_i) = \ell'$.

Conditions (1), (2) and (3) allow generating a sibling invariant lamination from a finite amount of initial data.

Full Sibling Collection ($d = 6$)



(Not to scale)

One of many possible sibling collections mapping to $\overline{xy}$.

Definition

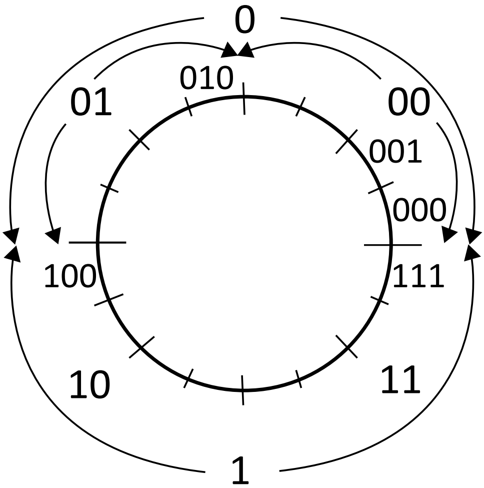
An orbit of polygons $P_0, P_1 = \sigma_d(P_0), P_2 = \sigma(P_1), \ldots$ is said to be *forward invariant* iff $\sigma_d : P_i \mapsto P_{i+1}$ preserves the circular order of the vertices of P_i.

Facts:

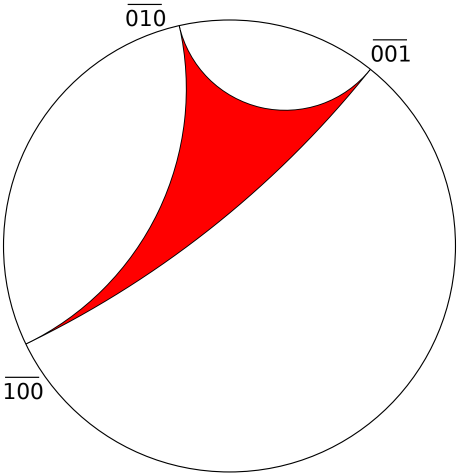
- If a finite orbit of polygons $P_0, P_1, P_2, \ldots, P_{n-1} = P_0$ is forward invariant under σ_2, then there always is a compatible critical chord touching the orbit at a vertex.
- If a finite orbit of polygons $P_0, P_1, P_2, \ldots, P_{n-1} = P_0$ is forward invariant under σ_3, then there are always two compatible critical chords touching the orbit at vertices.

(The facts can be generalized to a finite collection of finite orbits of polygons.)

σ_2 Binary Coordinates



Forward Invariant Triangle

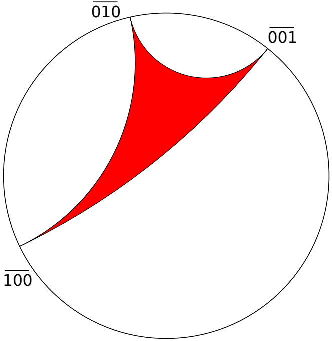


$$\sigma_2 : \quad \overline{001} \mapsto \overline{010} \mapsto \overline{100}$$

Pullback Scheme

Definition (Pullback Scheme)

A *pullback scheme* for σ_d is a collection of d branches $\tau_1, \tau_2, \ldots, \tau_d$ of the inverse of σ_d whose ranges partition $\partial\mathbb{D}$.

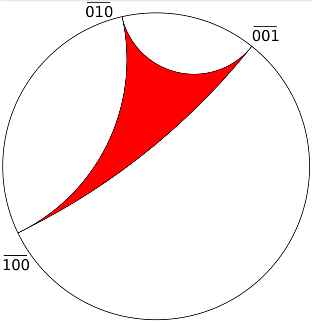


Data: Forward invariant lamination.

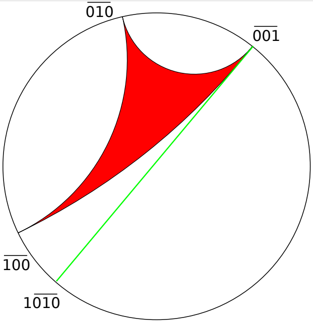
Pullback Scheme

Definition (Guiding Critical Chords)

The generating data of a pullback scheme are a *forward invariant periodic collection of leaves* and a collection of *d* interior disjoint *guiding critical chords*.

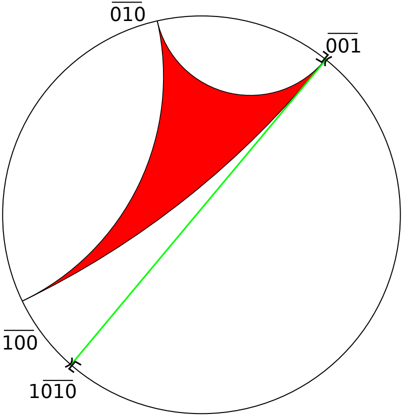


Data: Forward invariant lamination.



Guiding critical chord(s).

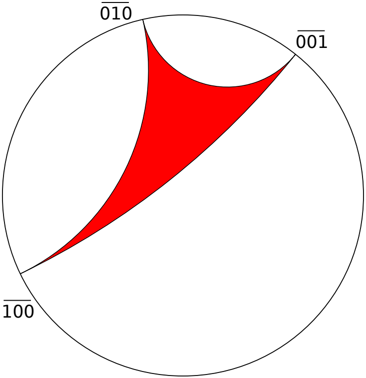
Branches τ_1, τ_2 of Inverse of σ_2



$\tau_1 : \partial\mathbb{D} \to [\overline{001}, 1\overline{010})$

$\tau_2 : \partial\mathbb{D} \to [1\overline{010}, \overline{001})$

Pullback Scheme



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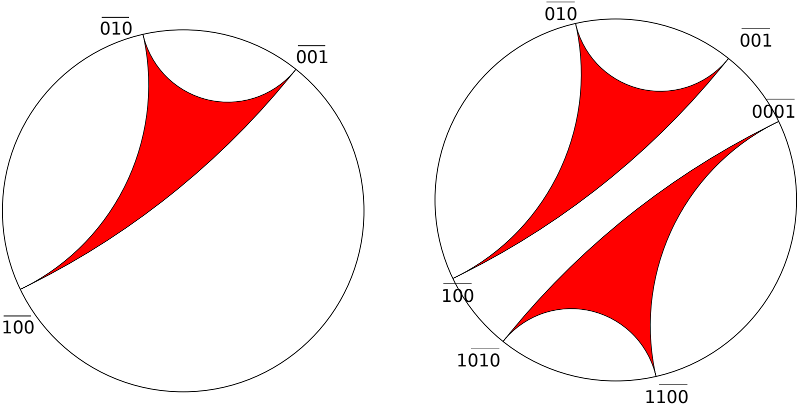
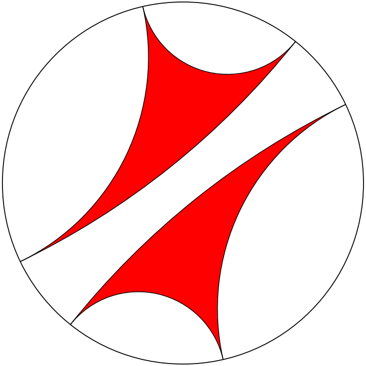


$$\sigma_2 : \quad 1\overline{010}, 0\overline{010} \mapsto \overline{010}$$

Pullback Scheme



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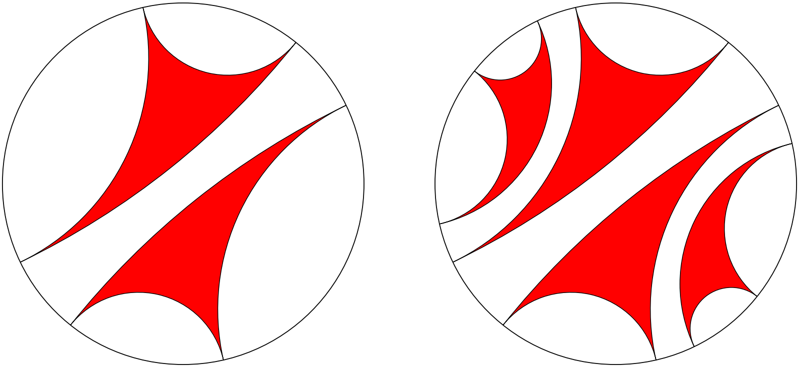

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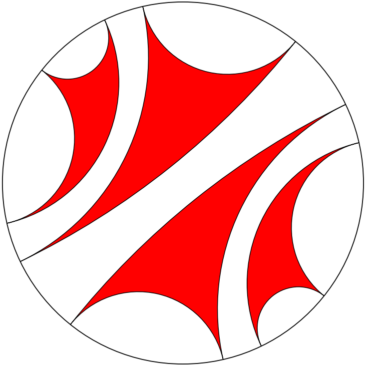

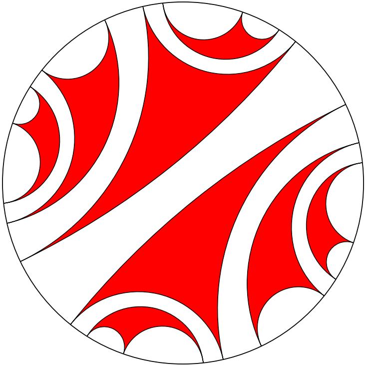

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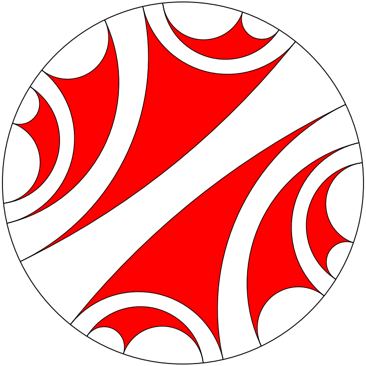

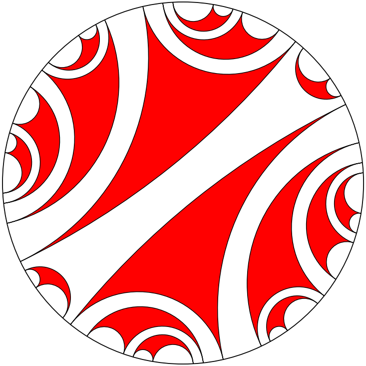

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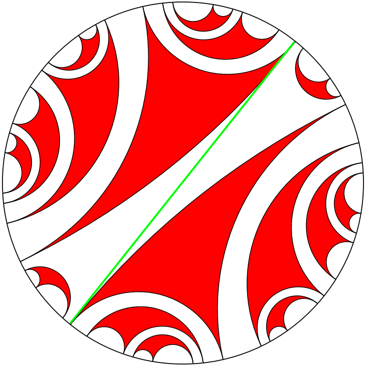
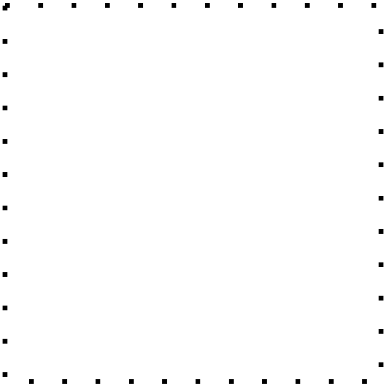
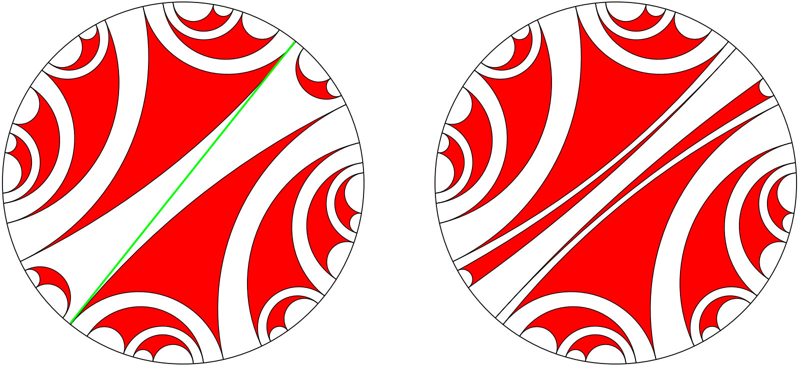


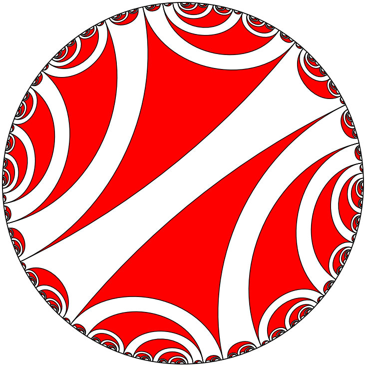


Ambiguity

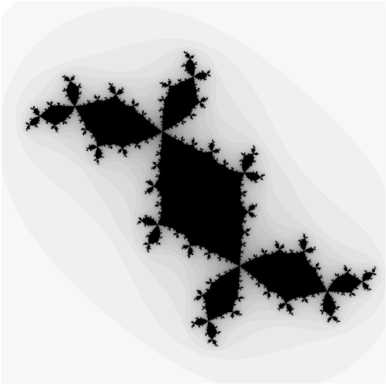


Quadratic Lamination and Julia Set

Rabbit Lamination



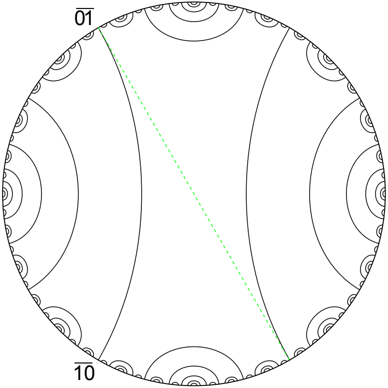
Rabbit Julia Set



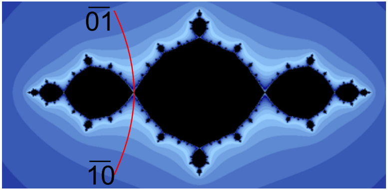
Quotient space in plane $\Longrightarrow$ homeomorphic to rabbit Julia set.

Quadratic Lamination and Julia Set

Basillica Lamination

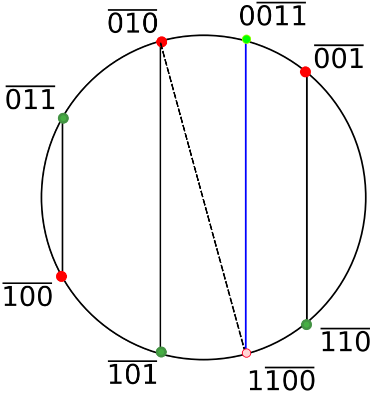


Basillica Julia Set

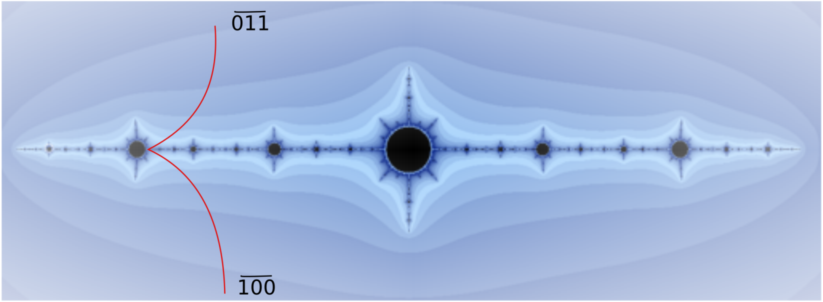


Identity Return Leaf Orbit for σ_2

$$\sigma_2 : [\overline{011}, \overline{100}] \mapsto [\overline{110}, \overline{001}] \mapsto [\overline{101}, \overline{010}]$$

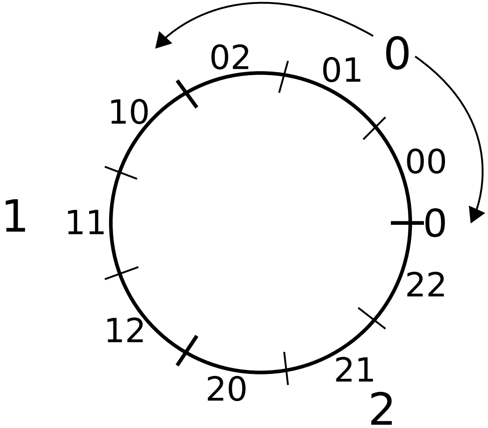


Airplane Quadratic Julia Set



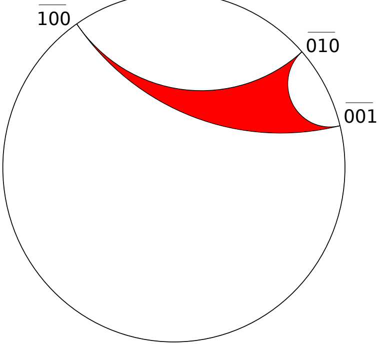
The corresponding point in the Julia set has two ray orbits landing on it.

σ_3 ternary coordinates



Cubic Lamination and Julia Set

Cubic Rabbit Triangle



Cubic Lamination and Julia Set

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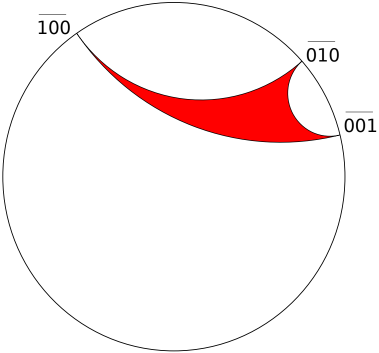
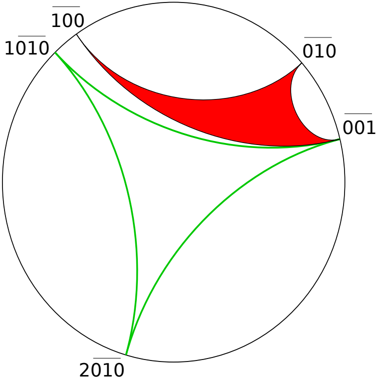
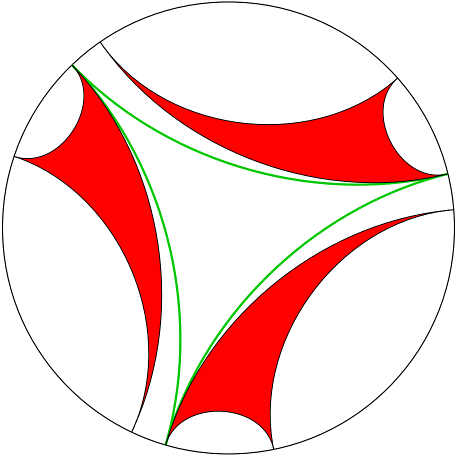


Guiding all-critical triangle



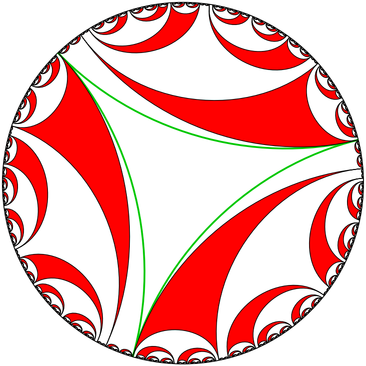
Cubic Lamination and Julia Set



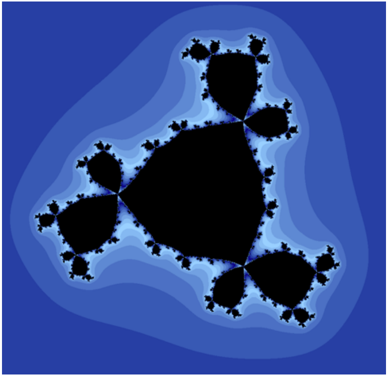
Symmetric Siblings

Cubic Lamination and Julia Set

Cubic Rabbit Lamination

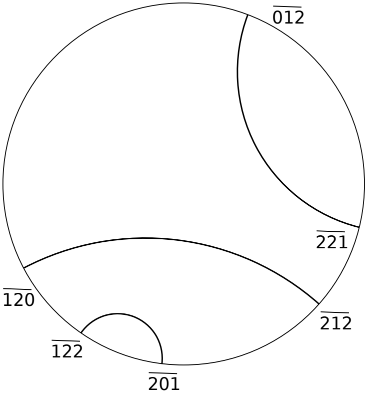


Cubic Rabbit Julia Set



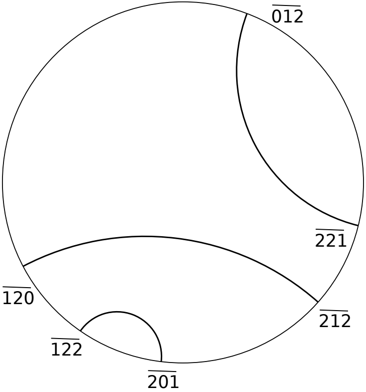
Cubic Pullback: Identity Return Leaf for σ_3

An *Identity Return Leaf* for σ_3.

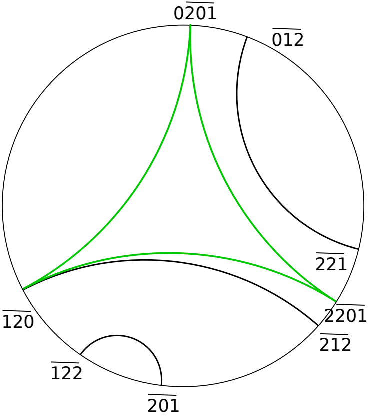


$$[\overline{120}, \overline{212}] \mapsto [\overline{201}, \overline{122}] \mapsto [\overline{012}, \overline{221}]$$

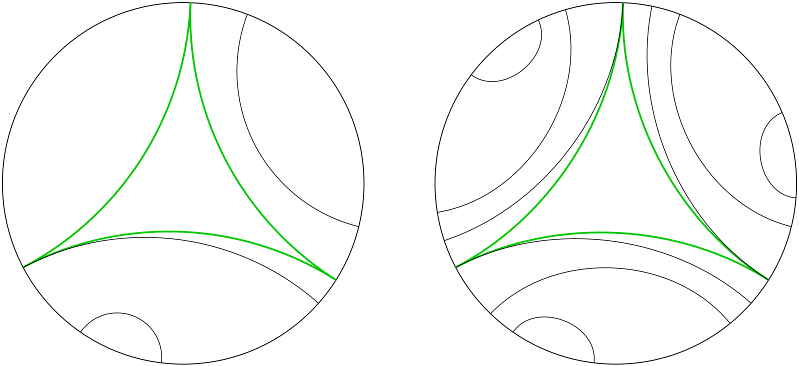
Identity Return Leaf for σ_3



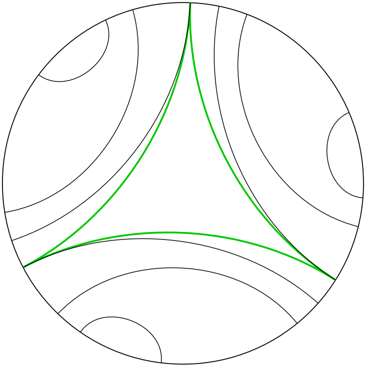
Orbit admits an all-critical triangle.

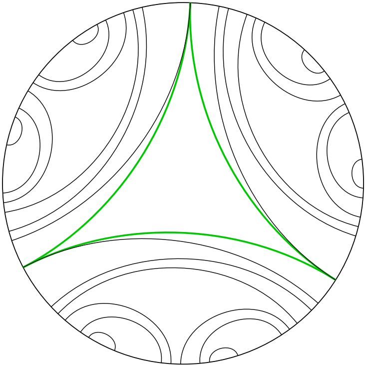


Identity Return Leaf for σ_3

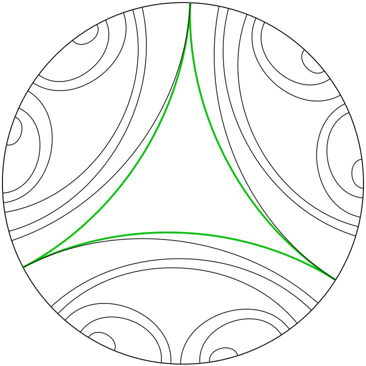


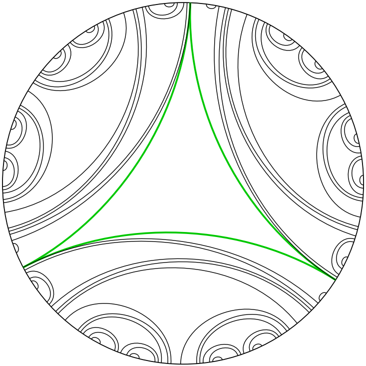
Identity Return Leaf for σ_3



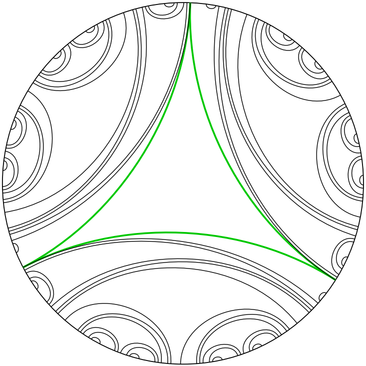


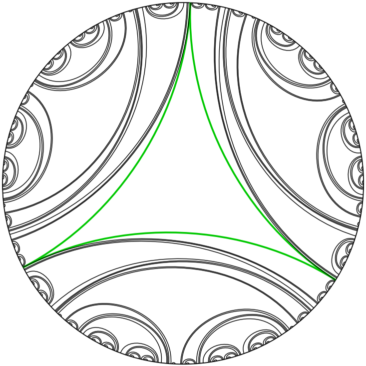
Identity Return Leaf for σ_3





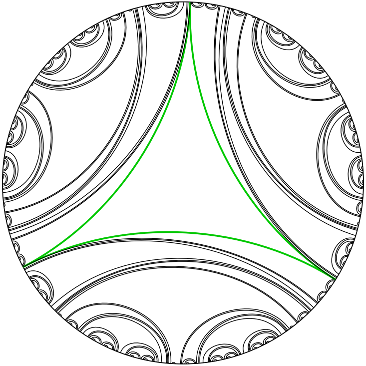
Identity Return Leaf for σ_3



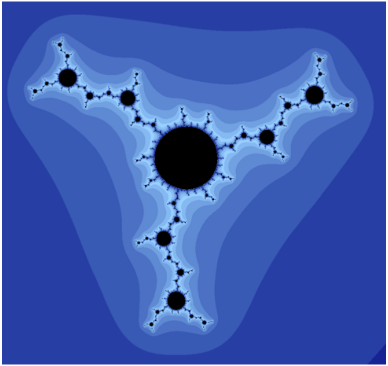


Cubic Lamination and Julia Set

Identity Return Leaf Lamination



Helicopter Julia Set



$$z \mapsto z^3 - 0.2634 - 1.2594i$$

Identity Return Polygons

Definition

A polygon $P = P_0$ is said to be *identity return* iff its *orbit*

$$\{P_0, P_1 = \sigma_d(P_0), P_2 = \sigma_d(P_1), P_3, \ldots, P_n = P_0\}$$

is periodic (of least period n) and has the properties

1. the polygons in the orbit are disjoint,
2. $\sigma_d^n|_{P_0}$ is the identity, and
3. P_i maps to $P_{i+1 \pmod n}$ preserving circular order.

- Each vertex is in a different orbit of period n.

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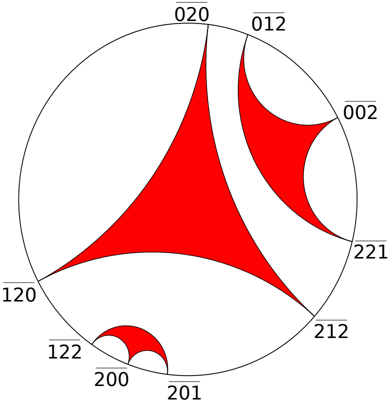
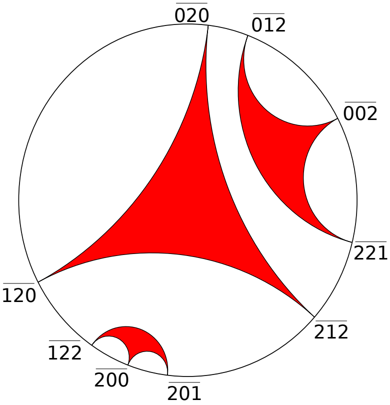
Cubic Pulback: Identity Return Triangle

Where can one place two critical chords to start the pullback process?

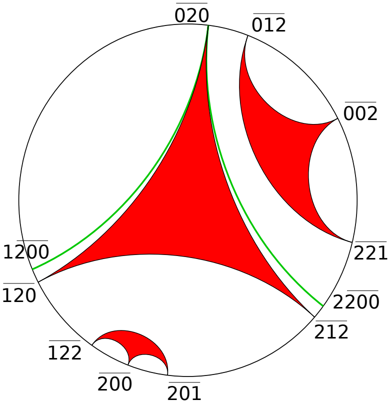


Forward invariant lamination given

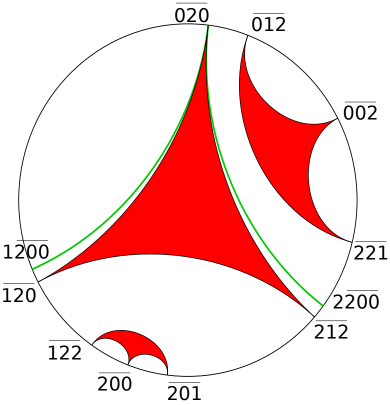
Identity Return Triangle

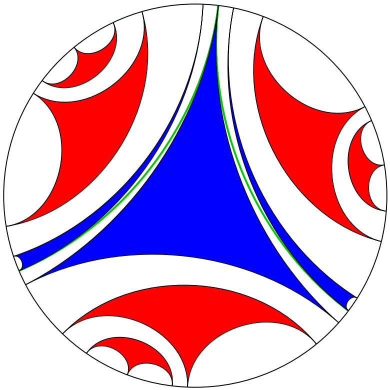


Guiding critical chords



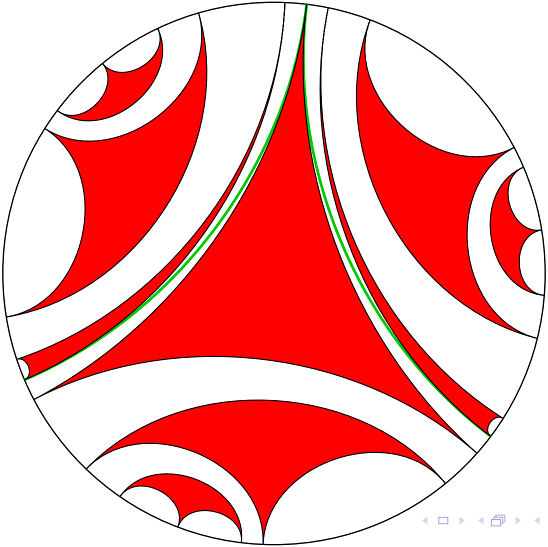
Identity Return Triangle



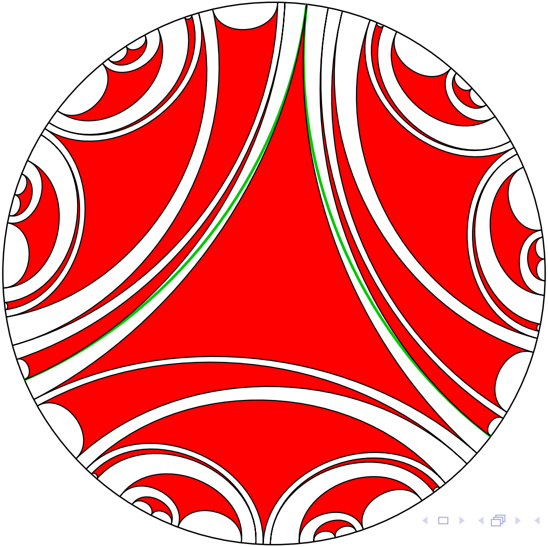


Non-symmetric siblings

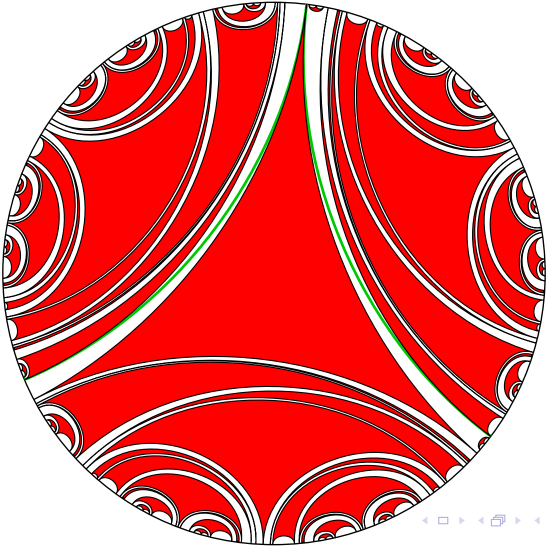
Identity Return Triangle



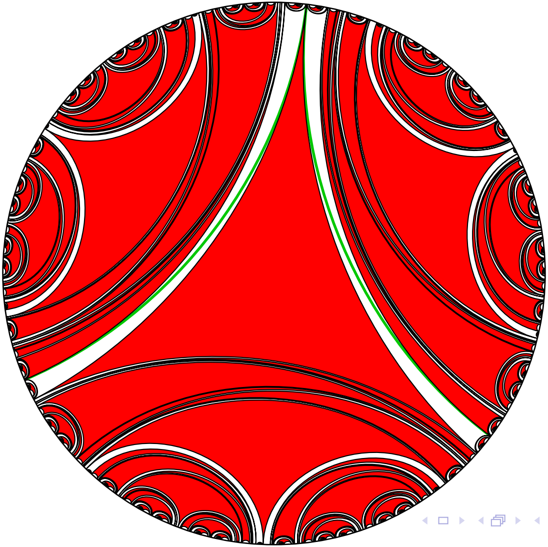
Identity Return Triangle



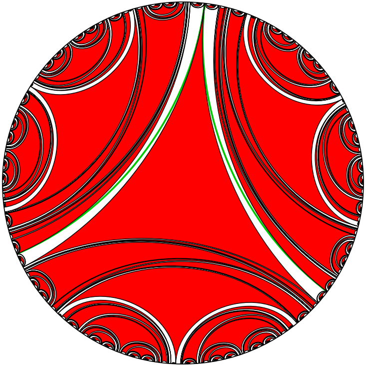
Identity Return Triangle

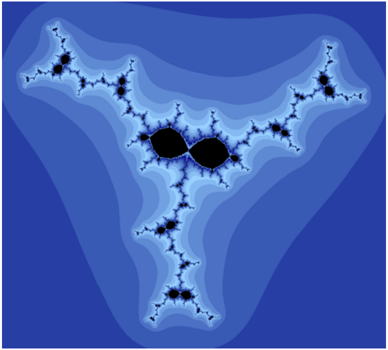


Identity Return Triangle



Identity Return Triangle and Corresponding Julia Set



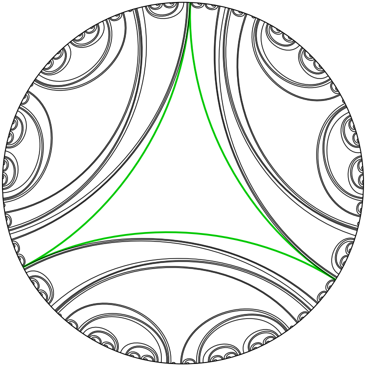


$$z \mapsto z^3 + 3fz^2 + g$$

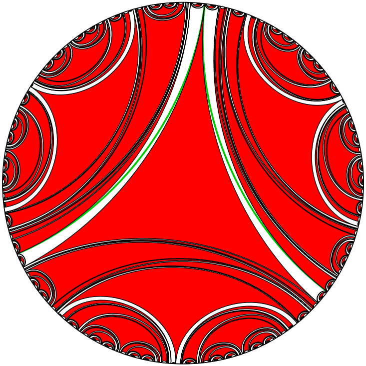
$f = -0.167026 + 0.0384441i$ and $g = -0.0916222 - 1.2734i$

Identity Return Leaf versus Identity Return Triangle

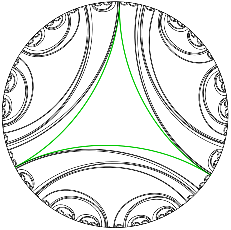
Identity Return Leaf $[\overline{120}, \overline{212}]$

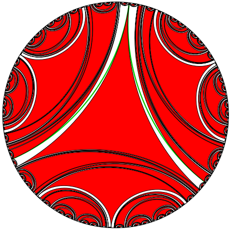


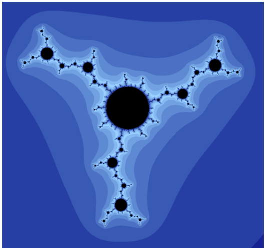
Identity Return Triangle with One Side $[\overline{120}, \overline{212}]$

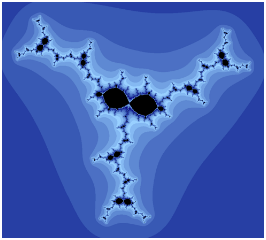


Comparison









Sampling of Questions

1. Under what circumstances can multiple Identity Return Polygon (IRP) orbits co-exist in an invariant lamination?
2. Given 3 points of a given period $p \geq 3$, what are the criteria for forming an Identity Return Triangle (IRT) for σ_3? [Brandon Barry – Dissertation]
3. In particular, can three given period p orbits form more than one IRT? [No – CHMMO]
4. Given $d \geq 2$ and a period $p > 1$ orbit under σ_d, how many distinct identity return d-gon orbits can be formed?
5. What is the "simplest" 3-invariant lamination that contains a given IRT? [Brandon Barry – Dissertation]
6. Given a "simplest" IRT lamination, is there a cubic Julia set for which it is the lamination?

References

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  Topology Proceedings 48 (2016), pp. 69–100. E-published on April 17, 2015.

- Mayer, J.C. and Mernik, L.
  Periodic Polygons in d-Invariant Laminations of the Unit Disk
  Submitted June 2016.

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  On the Simplest Lamination of a Given Identity Return Triangle
  Dissertation, UAB, July 2015.