On the Length of Borel Hierarchies

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The Borel hierachy is described as follows:

open $= \utilde{\mathbf{\Sigma}}^0_1 = G$

closed $= \utilde{\mathbf{\Pi}}^0_1 = F$

$\utilde{\mathbf{\Pi}}^0_2 = G_\delta =$ countable intersections of open sets

$\utilde{\mathbf{\Sigma}}^0_2 = F_\sigma =$ countable unions of closed sets

$\utilde{\mathbf{\Sigma}}^0_\alpha = \{ \bigcup_{n<\omega} A_n \ : \ A_n \in \utilde{\mathbf{\Pi}}^0_{<\alpha} = \bigcup_{\beta<\alpha} \utilde{\mathbf{\Pi}}^0_\beta \}$

$\utilde{\mathbf{\Pi}}^0_\alpha =$ complements of $\utilde{\mathbf{\Sigma}}^0_\alpha$ sets

Borel $= \utilde{\mathbf{\Pi}}^0_{\omega_1} = \utilde{\mathbf{\Sigma}}^0_{\omega_1}$

In a metric space for $1 \leq \alpha < \beta$

$$\utilde{\mathbf{\Sigma}}^0_\alpha \cup \utilde{\mathbf{\Pi}}^0_\alpha \ \subseteq \ \utilde{\mathbf{\Sigma}}^0_\beta \cap \utilde{\mathbf{\Pi}}^0_\beta = \utilde{\mathbf{\Delta}}^0_\beta$$

Theorem (Lebesgue 1905)

For every countable $\alpha > 0$

$$\underset{\sim}{\mathbf{\Sigma}}^0_\alpha(2^\omega) \neq \underset{\sim}{\mathbf{\Pi}}^0_\alpha(2^\omega).$$

Define $\mathrm{ord}(X)$ to be the least α such that $\underset{\sim}{\mathbf{\Sigma}}^0_\alpha(X) = \underset{\sim}{\mathbf{\Pi}}^0_\alpha(X)$.

Hence $\mathrm{ord}(2^\omega) = \omega_1$. If X is any topological space which contains a homeomorphic copy of 2^ω, then $\mathrm{ord}(X) = \omega_1$. More generally, if $Y \subseteq X$, then $\mathrm{ord}(Y) \leq \mathrm{ord}(X)$.

If X countable, then $\mathrm{ord}(X) \leq 2$.

Theorem (Bing, Bledsoe, Mauldin 1974)

Suppose $(2^\omega, \tau)$ *is second countable and refines the usual topology. Then* $\mathrm{ord}(2^\omega, \tau) = \omega_1$.

Theorem (Recław 1993)

If X is a second countable space and X can be mapped continuously onto any space containing 2^ω, then $\mathrm{ord}(X) = \omega_1$.

Q. It is consistent that for any $2 \le \beta \le \omega_1$ there are $X, Y \subseteq 2^\omega$ and $f : X \to Y$ continuous, one-to-one, and onto such that $\mathrm{ord}(X) = 2$ and $\mathrm{ord}(Y) = \beta$. What other pairs of orders (α, β) are possible?

Corollary

If X is separable, metric, but not zero-dimensional, then $\mathrm{ord}(X) = \omega_1$.

If X is separable, metric, and zero-dimensional, then it is homeomorphic to a subspace of 2^ω.

Theorem (Poprougenko and Sierpiński 1930)

If $X \subseteq 2^\omega$ is a Luzin set, then $\mathrm{ord}(X) = 3$.

If $X \subseteq 2^\omega$ is a Luzin set, then for every Borel set B there are $\utilde{\mathbf{\Pi}}^0_2$ C and $\utilde{\mathbf{\Sigma}}^0_2$ D such that $B \cap X = (C \cup D) \cap X$.

Q. Can we have $X \subseteq 2^\omega$ with $\mathrm{ord}(X) = 4$ and for every Borel set B there are $\utilde{\mathbf{\Pi}}^0_3$ C and $\utilde{\mathbf{\Sigma}}^0_3$ D such that $B \cap X = (C \cup D) \cap X$?

Q. Same question for the β^{th} level of the Hausdorff difference hierarchy inside the $\utilde{\mathbf{\Delta}}^0_{\alpha+1}$ sets?

Theorem (Szpilrajn 1930)

If $X \subseteq 2^\omega$ is a Sierpiński set, then $\mathrm{ord}(X) = 2$.

Theorem (Miller 1979)

The following are each consistent with ZFC:

- *for all* $\alpha < \omega_1$ *there is* $X \subseteq 2^\omega$ *with* $\mathrm{ord}(X) = \alpha$.
- $\mathit{ord}(X) = \omega_1$ *for all uncountable* $X \subseteq 2^\omega$.
- $\{\alpha : \alpha_0 < \alpha \leq \omega_1\} = \{\mathrm{ord}(X) : \textit{ unctbl } X \subseteq 2^\omega\}$.

Q. What other sets can $\{\mathrm{ord}(X) : \text{ unctbl } X \subseteq 2^\omega\}$ be?
$\{\alpha : \omega \leq \alpha \leq \omega_1\}$? Even ordinals?

Theorem (Miller 1979)

For any $\alpha \leq \omega_1$ there is a complete ccc Boolean algebra $\mathbb{B}$ which can be countably generated in exactly α steps.

Theorem (Kunen 1979)

(CH) For any $\alpha < \omega_1$ there is an $X \subseteq 2^\omega$ with $\mathrm{ord}(X) = \alpha$.

Theorem (Fremlin 1982)

(MA) For any $\alpha < \omega_1$ there is an $X \subseteq 2^\omega$ with $\mathrm{ord}(X) = \alpha$.

Theorem (Miller 1979a)

For any α with $1 \leq \alpha < \omega_1$ there is a countable set G_α of generators of the category algebra, Borel(2^ω) mod meager, which take exactly α steps.

Cohen real model and Random real model

Theorem (Miller 1995)

If there is a Luzin set of size κ, then for any α with $3 \leq \alpha < \omega_1$ there is an $X \subseteq 2^\omega$ of size κ and hereditarily of order α.

In the Cohen real model there is $X, Y \in [2^\omega]^{\omega_1}$ with hereditary order 2 and ω_1 respectively. Also, every $X \in [2^\omega]^{\omega_2}$ has $\mathrm{ord}(X) \geq 3$ and contains $Y \in [X]^{\omega_2}$ with $\mathrm{ord}(Y) < \omega_1$.

Theorem (Miller 1995)

In the random real model, for any α with $2 \leq \alpha \leq \omega_1$ there is an $X_\alpha \subseteq 2^\omega$ of size ω_1 with $\alpha \leq \mathrm{ord}(X_\alpha) \leq \alpha + 1$.

Q. Presumably, $\mathrm{ord}(X_\alpha) = \alpha$ but I haven't been able to prove this.

Sacks real model

Theorem (Miller 1995)

In the iterated Sacks real model for any α with $2 \le \alpha \le \omega_1$ there is an $X \subseteq 2^\omega$ of size ω_1 with $\operatorname{ord}(X) = \alpha$. Every $X \subseteq 2^\omega$ of size ω_2 has order ω_1.

In this model there is a Luzin set of size ω_1.
Also for every $X \subseteq 2^\omega$ of size ω_2 there is a continuous onto map $f : X \to 2^\omega$ (Miller 1983) and hence by (Recław 1993) $\operatorname{ord}(X) = \omega_1$.

What if Axiom of Choice fails

Theorem (Miller 2008)

It is consistent with ZF that $\mathrm{ord}(2^\omega) = \omega_2$.

This implies that ω_1 has countable cofinality, so the axiom of choice fails very badly in our model. We also show that using Gitik's model (1980) where every cardinal has countable cofinality, there are models of ZF in which the Borel hierarchy is arbitrarily long. It cannot be "class" long.

Q. If we change the definition of $\underset{\sim}{\mathbf{\Sigma}}^0_\alpha$ so that it is closed under countable unions, then I don't know if the Borel hierarchy can have length greater than ω_1.

Q. Over a model of ZF can forcing with *Fin*$(\kappa, 2)$ collapse cardinals?
(Martin Goldstern: No)

The levels of the ω_1-Borel hierarchy of subsets of 2^ω

- $\mathbf{\Sigma}^*_0 = \mathbf{\Pi}^*_0 =$ clopen subsets of 2^ω
- $\mathbf{\Sigma}^*_\alpha = \{\cup_{\beta<\omega_1} A_\beta \ : \ (A_\beta : \beta < \omega_1) \in (\cup_{\beta<\alpha}\mathbf{\Pi}^*_\beta)^{\omega_1}\}$
- $\mathbf{\Pi}^*_\alpha = \{2^\omega \setminus A \ : \ A \in \mathbf{\Sigma}^*_\alpha\}$

CH $\to \mathbf{\Pi}^*_2 = \mathbf{\Sigma}^*_2 = \mathcal{P}(2^\omega)$

Theorem (Miller 2011)

(MA+notCH) $\quad \mathbf{\Pi}^*_\alpha \neq \mathbf{\Sigma}^*_\alpha$ *for every positive* $\alpha < \omega_2$.

Q. What about the $< \mathfrak{c}$-Borel hierarchy for $\mathfrak{c}$ weakly inaccessible?

Theorem (Miller 2011)

In the Cohen real model $\mathbf{\Sigma}^*_{\omega_1+1} = \mathbf{\Pi}^*_{\omega_1+1}$ *and* $\mathbf{\Sigma}^*_\alpha \neq \mathbf{\Pi}^*_\alpha$ *for every* $\alpha < \omega_1$.

Q. I don't know if $\mathbf{\Sigma}^*_{\omega_1} = \mathbf{\Pi}^*_{\omega_1}$.

Q. (Brendle, Larson, Todorcevic 2008) Is it consistent with notCH to have $\mathbf{\Pi}^*_2 = \mathbf{\Sigma}^*_2$?

Theorem (Steprans 1982)

It is consistent that $\mathbf{\Pi}^*_3 = \mathbf{\Sigma}^*_3 = \mathcal{P}(2^\omega)$ *and* $\mathbf{\Pi}^*_2 \neq \mathbf{\Sigma}^*_2$.

Theorem (Carlson 1982)

If every subset of 2^ω *is* ω_1*-Borel, then the cofinality of the continuum must be* ω_1.

Theorem (Miller 2012)

(1) If $\mathcal{P}(2^\omega) = \omega_1$*-Borel, then* $\mathcal{P}(2^\omega) = \mathbf{\Sigma}^*_\alpha$ *for some* $\alpha < \omega_2$.
(2) For each $\alpha \leq \omega_1$ *it is consistent that* $\mathbf{\Sigma}^*_{\alpha+1} = \mathcal{P}(2^\omega)$ *and* $\mathbf{\Sigma}^*_{<\alpha} \neq \mathcal{P}(2^\omega)$*, i.e. length* α *or* $\alpha + 1$.

Q. Can it have length α for some α with $\omega_1 < \alpha < \omega_2$?

$X \subseteq 2^\omega$ is a Q_α-set iff $\mathrm{ord}(X) = \alpha$ and $\mathrm{Borel}(X) = \mathcal{P}(X)$.
Q-set is the same as Q_2-set.

Theorem (Fleissner, Miller 1980)

It is consistent to have an uncountable Q-set $X \subseteq 2^\omega$ which is concentrated on $E = \{x \in 2^\omega \ : \ \forall^\infty n \ \ x(n) = 0\}$. Hence $X \cup E$ is a Q_3-set.

Theorem (Miller 2014)

(CH) For any α with $3 \leq \alpha < \omega_1$ there are $X_0, X_1 \subseteq 2^\omega$ with $\mathrm{ord}(X_0) = \alpha = \mathrm{ord}(X_1)$ and $\mathrm{ord}(X_0 \cup X_1) = \alpha + 1$.

Q. Is it consistent that the X_i be Q_α-sets?
Q. What about getting $\mathrm{ord}(X_0 \cup X_1) \geq \alpha + 2$?

Theorem (Judah and Shelah 1991)

It is consistent to have a Q-set and $\mathfrak{d} = \omega_1$ using an iteration of proper forcings with the Sacks property.

Q. What about a Q_α-set for $\alpha > 2$?

Theorem (Miller 2003)

It is consistent to have a Q-set $X \subseteq [\omega]^\omega$ which is a maximal almost disjoint family.

Q. It is consistent to have Q-set $\{x_\alpha : \alpha < \omega_1\}$ and a non Q-set $\{y_\alpha : \alpha < \omega_1\}$ such that $x_\alpha =^* y_\alpha$ all α.
Can $\{x_\alpha : \alpha < \omega_1\}$ be MAD?

Products of Q-sets

Theorem (Brendle 2016)

It is consistent to have a Q-set X such that X^2 is not a Q-set.

Theorem (Miller 2016)

(1) If X^2 Q_α-set and $|X| = \omega_1$, then X^n is a Q_α-set for all n.
(2) If X^3 Q_α-set and $|X| = \omega_2$, then X^n is a Q_α-set for all n.
(3) If $|X_i| < \omega_n$, $\prod_{i\in K} X_i$ a Q_α-set for every $K \in [N]^n$, then $\prod_{i\in N} X_i$ is a Q_α-set.

Q. Can we have X^2 a Q-set and X^3 not a Q-set?
Q. For $\alpha > 2$ can we have X a Q_α-set but X^2 not a Q_α-set?

Theorem (Miller 1995)

(CH) For any α with $3 \leq \alpha < \omega_1$ there is a $Y \subseteq 2^\omega$ such that $\mathrm{ord}(Y) = \alpha$ and $\mathrm{ord}(Y^2) = \omega_1$.

Q. Can we have $\alpha < \mathrm{ord}(Y^2) < \omega_1$?

Theorem (Miller 1979)

If $\mathrm{Borel}(X) = \mathcal{P}(X)$*, then* $\mathrm{ord}(X) < \omega_1$*. There is no* Q_{ω_1}*-set.*

Theorem (Miller 1979)

It is consistent to have: for every $\alpha < \omega_1$ *there is a* Q_α*-set.*

In this model the continuum is $\aleph_{\omega_1+1}$.

Q. For $\alpha \geq 3$ can we have a Q_α of cardinality greater than or equal to some $Q_{\alpha+1}$-set?

Q. If we have a Q_ω-set must there be Q_n-sets for inf many $n < \omega$?

Q. Can there be a Q_ω-set of cardinality ω_1?

Theorem (Miller 1979, 2014)

For any successor α with $3 \leq \alpha < \omega_1$ it is consistent to have a Q_α-set but no Q_β-set for $\beta < \alpha$.

In this model the continuum has cardinality ω_2. The Q_α-set X has size ω_1 and has "strong order" α. Namely, even if you add countably many more sets to the topology of X its order is still α. Another way to say this is that in this model $\mathcal{P}(\omega_1)$ is a countably generated σ-algebra in α-steps but it cannot be countably generated in fewer steps. (In fact, not even with ω_1-generators.) I don't know about Q_β sets for $\beta > \alpha$. However if Brendle's argument can be generalized it would show that X^2 is a $Q_{\alpha+1}$-set.

Abstract Rectangles

Theorem (Rao 1969, Kunen 1968)

Assume the continuum hypothesis then every subset of the plane is in the σ-algebra generated by the abstract rectangles at level 2:

$$\mathcal{P}(2^\omega \times 2^\omega) = \sigma_2(\{A \times B \ : \ A, B \subseteq 2^\omega\}).$$

Theorem (Kunen 1968)

Assume Martin's axiom, then

$$\mathcal{P}(2^\omega \times 2^\omega) = \sigma_2(\{A \times B \ : \ A, B \subseteq 2^\omega\}).$$

In the Cohen real model or the random real model any well-ordering of 2^ω is not in the σ-algebra generated by the abstract rectangles.

Theorem (Rothberger 1952 Bing, Bledsoe, Mauldin 1974)

Suppose that $2^\omega = \omega_2$ and $2^{\omega_1} = \aleph_{\omega_2}$ then the σ-algebra generated by the abstract rectangles in the plane is not the power set of the plane.

Theorem (Bing, Bledsoe, Mauldin 1974)

If every subset of the plane is in the σ-algebra generated by the abstract rectangles, then for some countable α every subset of the plane is in the σ-algebra generated by the abstract rectangles by level α. $\mathcal{P}(2^\omega \times 2^\omega) = \sigma_\alpha(\{A \times B : A, B \subseteq 2^\omega\})$

Theorem (Miller 1979)

For any countable $\alpha \geq 2$ it is consistent that every subset of the plane is in the σ-algebra generated by the abstract rectangles at level α but for every $\beta < \alpha$ not every subset is at level β.
$\mathrm{ord}(\sigma(\{A \times B : A, B \subseteq 2^\omega\})) = \alpha$

Theorem (Miller 1979)

Suppose $2^{<\mathfrak{c}} = \mathfrak{c}$ *and* $\alpha < \omega_1$*. Then the following are equivalent:*
(1) Every subset of $2^\omega \times 2^\omega$ *is in the* σ*-algebra generated by the abstract rectangles at level* α*.*
(2) There exists $X \subseteq 2^\omega$ *with* $|X| = \mathfrak{c}$ *and every subset of* X *of cardinality less than* $\mathfrak{c}$ *is* Σ^0_α *in* X*.*

Q. Can we have $2^{<\mathfrak{c}} \neq \mathfrak{c}$ and $\mathcal{P}(2^\omega \times 2^\omega) = \sigma(\{A \times B : A, B \subseteq 2^\omega\})$?

Q. Can we have $\mathrm{ord}(\sigma(\{A \times B : A, B \subseteq 2^\omega\})) < \omega_1$ and $\mathcal{P}(2^\omega \times 2^\omega) \neq \sigma(\{A \times B : A, B \subseteq 2^\omega\})$?

Q. Can we have $\mathrm{ord}(\sigma(\{A \times B : A, B \subseteq 2^\omega\}))$ be strictly smaller than $\mathrm{ord}(\sigma(\{A \times B \times C : A, B, C \subseteq 2^\omega\}))$?

Theorem (Larson, Miller, Steprans, Weiss 2014)

Suppose $2^{<\mathfrak{c}} = \mathfrak{c}$ then the following are equivalent:
(1) There is a Borel universal function, i.e, a Borel function $F : 2^\omega \times 2^\omega \to 2^\omega$ such that for every abstract $G : 2^\omega \times 2^\omega \to 2^\omega$ there are $h : 2^\omega \to 2^\omega$ and $k : 2^\omega \to 2^\omega$ such that for every $x, y \in 2^\omega \quad G(x, y) = F(h(x), k(y))$.
(2) Every subset of the plane is in the σ-algebra generated by the abstract rectangles.

Furthermore the universal function has level α iff every subset of the plane is in the σ-algebra generated by the abstract rectangles at level α.

Abstract Universal Functions

Theorem (Larson, Miller, Steprans, Weiss 2014)

If $2^{<\kappa} = \kappa$, then there is an abstract universal function $F : \kappa \times \kappa \to \kappa$, i.e., $\forall G \exists h, k \forall \alpha, \beta \;\; G(\alpha, \beta) = F(h(\alpha), k(\beta))$.

Theorem (Larson, Miller, Steprans, Weiss 2014)

It is consistent that there is no abstract universal function $F : \mathfrak{c} \times \mathfrak{c} \to \mathfrak{c}$.

Q. Is it consistent with $2^{<\mathfrak{c}} \neq \mathfrak{c}$ to have a universal $F : \mathfrak{c} \times \mathfrak{c} \to \mathfrak{c}$?

Higher Dimensional Abstract Universal Functions

Theorem (Larson, Miller, Steprans, Weiss 2014)

Abstract universal functions $F : \kappa^n \to \kappa$ of higher dimensions reduce to countably many cases where the only thing that matters is the arity of the parameter functions, e.g.
(1) $\exists F \forall G \exists h, k \forall x, y \quad G(x, y) = F(h(x), k(y))$
(2) ...$G(x, y, z) = F(h(x, y), k(y, z), l(x, z))$
(n) ...$G(x_0, \ldots, x_n) = F(h_S(x_S) : S \in [n+1]^n)$

Theorem (Larson, Miller, Steprans, Weiss 2014)

In the Cohen real model for every $n \geq 1$ there is a universal function on ω_n where the parameter functions have arity $n+1$ but no universal function where the parameters functions have arity n.

A set is Souslin in X iff it has the form $\bigcup_{f\in\omega^\omega}\bigcap_{n<\omega} A_{f\restriction n}$ where each A_s for $s\in\omega^{<\omega}$ is Borel in X.

Theorem (Miller 1981)

It is consistent to have $X\subseteq 2^\omega$ such that every subset of X is Souslin in X and $\mathrm{ord}(X)=\omega_1$. A $Q_{\mathcal{S}}$-set.

Theorem (Miller 1995)

(CH) For any α with $2\leq\alpha\leq\omega_1$ there is exists an uncountable $X\subseteq 2^\omega$ such that $\mathrm{ord}(X)=\alpha$ and every Souslin set in X is Borel in X.

Theorem (Miller 2005)

It is consistent that there exists $X\subseteq 2^\omega$ such $\mathrm{ord}(X)\leq 3$ and there is a Souslin set in X which is not Borel in X.

Q. Can we have $\mathrm{ord}(X)=2$ here?

Theorem (Miller 1981)

It is consistent that for every subset $A \subseteq 2^\omega \times 2^\omega$ there are abstract rectangles $B_s \times C_s$ with $A = \bigcup_{f \in \omega^\omega} \bigcap_{n<\omega} (B_{f\restriction n} \times C_{f\restriction n})$ but $\mathcal{P}(2^\omega \times 2^\omega) \neq \sigma(\{A \times B : A, B \subseteq 2^\omega\})$.

Theorem (Miller 1979)

It is consistent that the universal Σ^1_1 set $U \subseteq 2^\omega \times 2^\omega$ is not in the σ-algebra generated by the abstract rectangles.

Theorem (Miller 1981)

It is consistent that there is no countably generated σ-algebra which contains all Σ^1_1 subsets of 2^ω.

Q. Thm 3 is stronger than Thm 2. Is the converse true?

For X a separable metric space define:

- $\Sigma_0^X = \Pi_0^X =$ Borel subsets of X^m some m.
- Σ_{n+1}^X the projections of Π_n^X sets.
- Π_{n+1}^X the complements of Σ_{n+1}^X sets.
- $\mathrm{Proj}(X) = \bigcup_{n<\omega} \Sigma_n^X$.

Theorem (Miller 1990)

It is consistent there are $X, Y, Z \subseteq 2^\omega$ of projective orders 0, 1, 2:

- $\mathrm{ord}(X) = \omega_1$ *and* $\Sigma_0^X = \mathit{Proj}(X)$
- $\Sigma_0^Y \neq \Sigma_1^Y = \mathit{Proj}(Y)$
- $\Sigma_0^Z \neq \Sigma_1^Z \neq \Sigma_2^Z = \mathit{Proj}(Z)$

Q. (Ulam) What about projective order 3 or higher?

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