# The space of invariant geometric laminations of degree $d$

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# Motivation

We consider polynomials in one complex variable

$$P(z) = a_0 + a_1 z + \cdots + a_{d-2} z^{d-2} + z^d$$

as topological dynamical systems on $\mathbb{C}$.
Parameter space is $\mathbb{C}^{d-1}$.

Any quadratic polynomial is affinely conjugate to

$$P_c(z) = z^2 + c.$$

# The basin of infinity and the Julia set

**Definition**

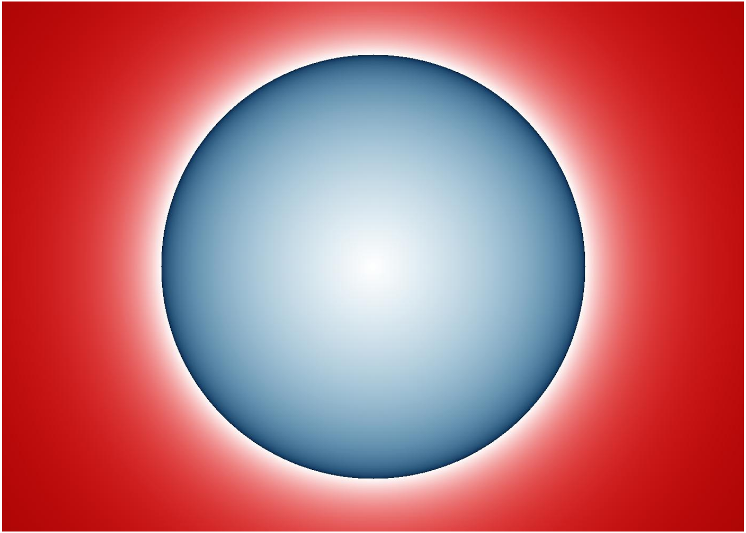
The basin of infinity

$$\Omega_P = \{z \in \mathbb{C} \mid P^{\circ n}(z) \to \infty \ (n \to \infty)\}$$

**Definition**

The Julia set $J(P) = \partial\Omega_P$ is an invariant set. The dynamics of $P|_{J(P)}$ is chaotic (not stable).

## $P_0(z) = z^2 + 0$



A super-attracting fixed point

Denote by $\sigma_d : \mathbb{S} \to \mathbb{S}$ the degree $d$ covering map of the unit circle defined by

$$\sigma_d(z) = z^d$$

We will parameterize the unit circle $\mathbb{S} = \mathbb{R}/\mathbb{Z}$ and, if we use this parameterization,

$$\sigma_d(t) = dt \mod 1$$

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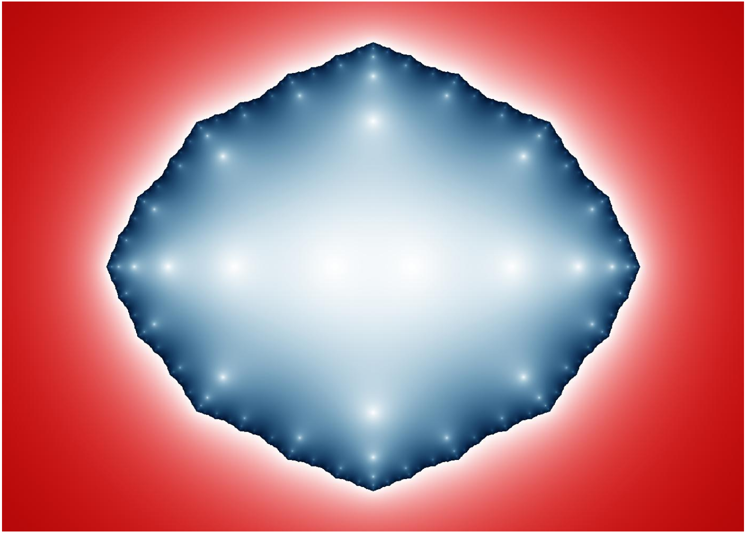
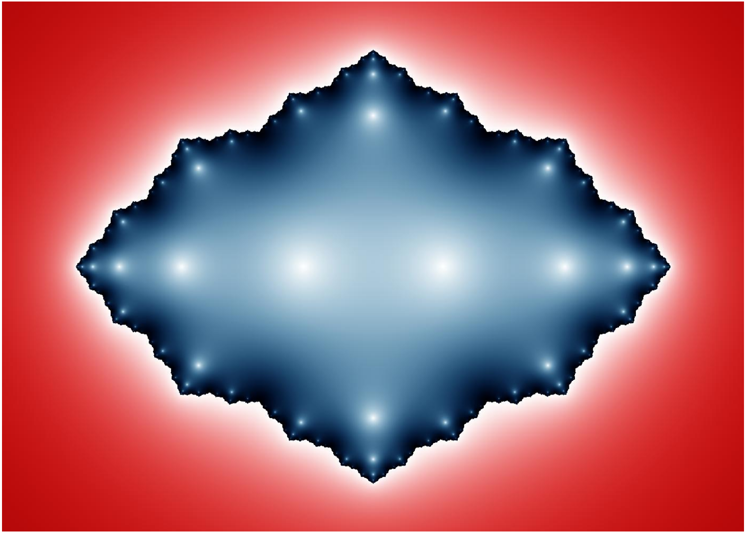
## $P(z) = z^2 - 0.2$



An attracting fixed point

## $P(z) = z^2 - 0.4$



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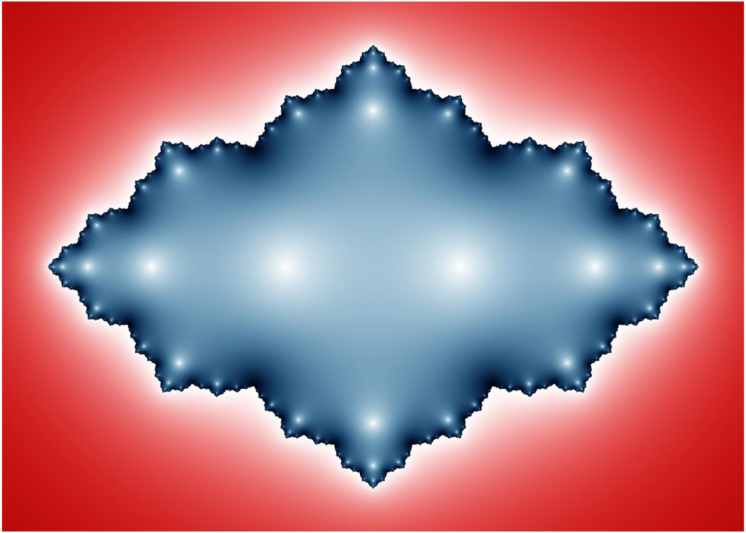
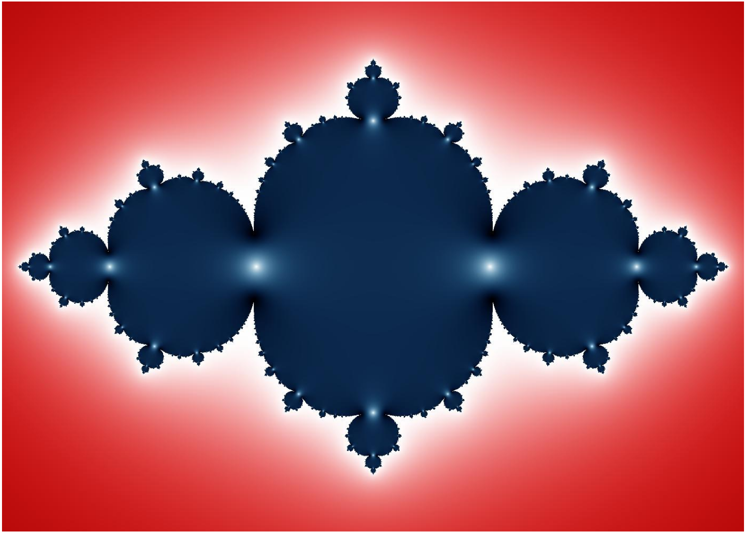
## $P(z) = z^2 - 0.5$



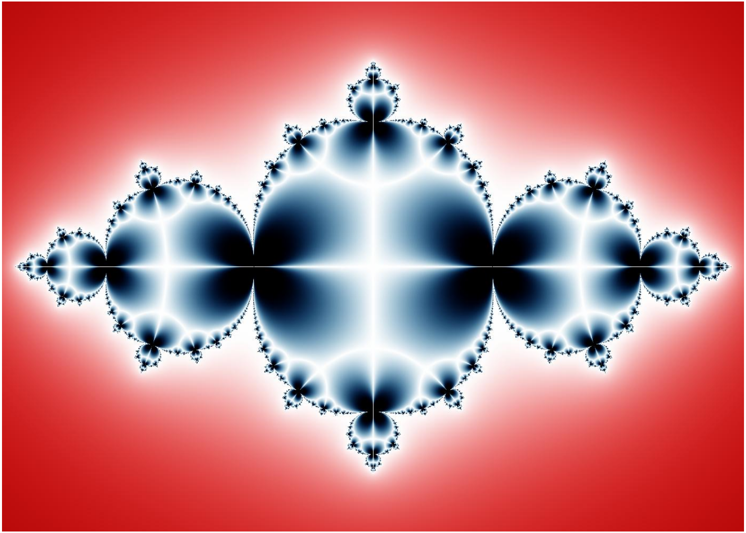
An attracting fixed point

$$P(z) = z^2 - 0.73$$



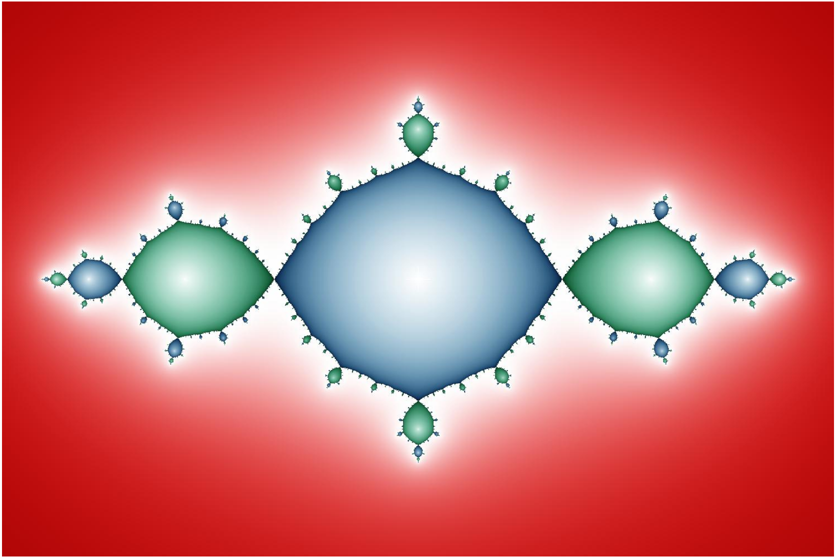
An attracting fixed point

# Parabolic bifurcation: $P(z) = z^2 - 0.75$



A fixed point with multiplier $-1$

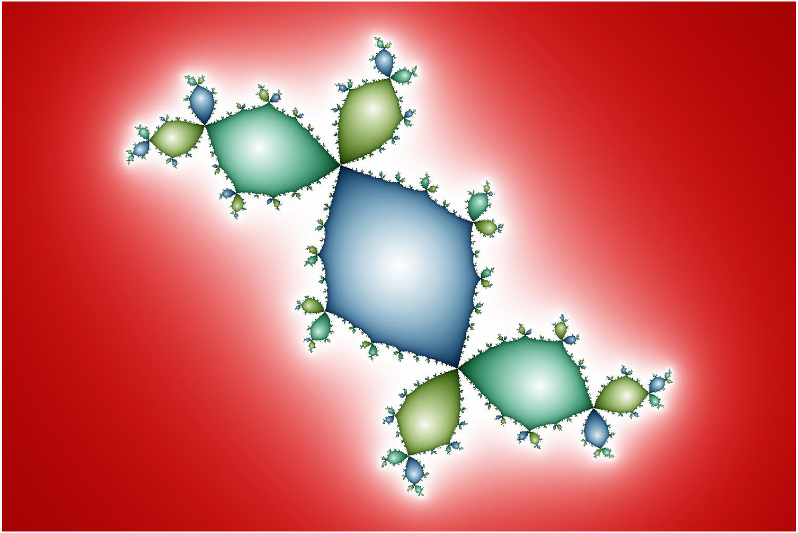
# Basilica: $P(z) = z^2 - 1$



A superattracting cycle of period 2

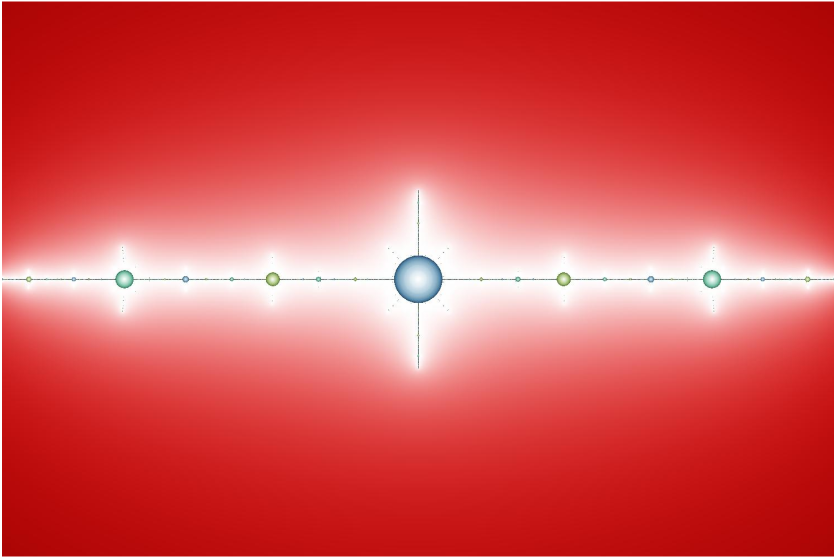
# Rabbit

# Douady rabbit: $f(z) = z^2 - 0.12.. + 0.74..i$



A superattracting cycle of period 3

# Airplane: $f(z) = z^2 - 1.75..$



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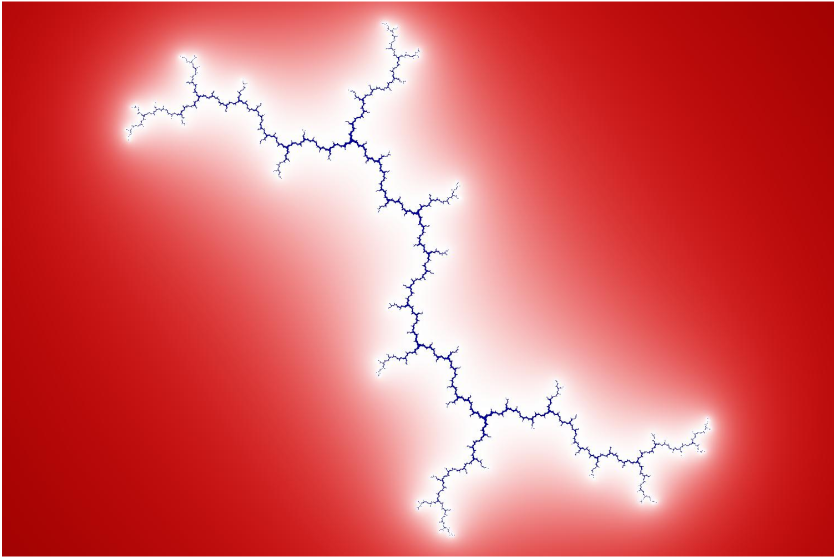
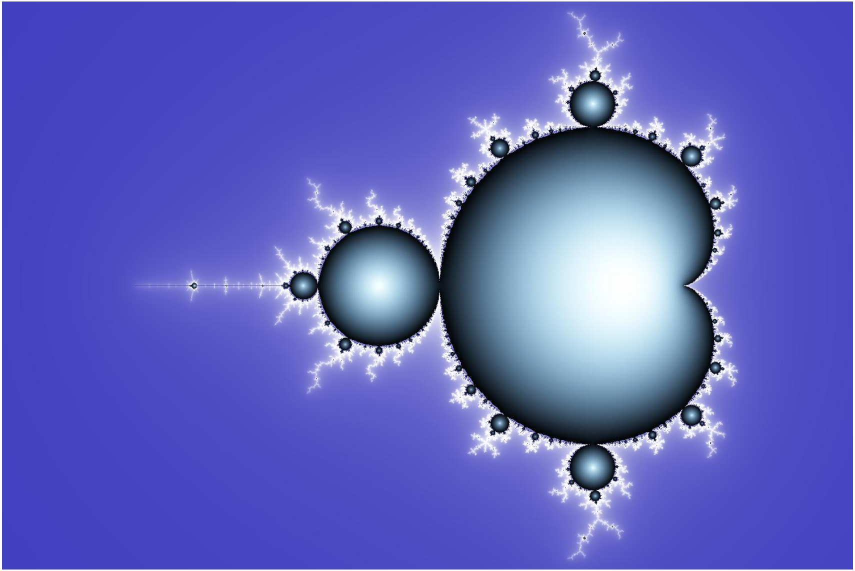
# Dendrite: $f(z) = z^2 + i$



All cycles are repelling

# The Mandelbrot set



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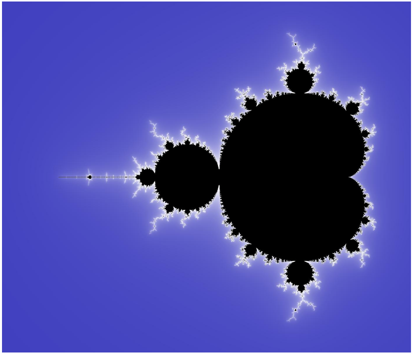
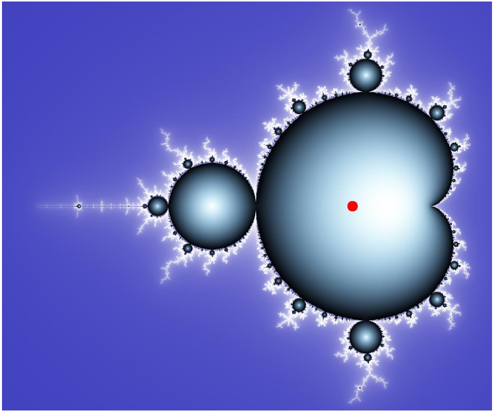


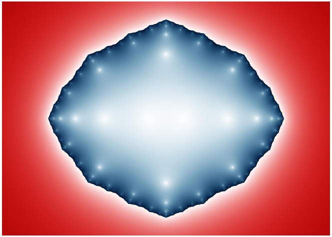
$$\mathcal{M}_2 = \{c \in \mathbb{C} \mid \textit{the sequence}$$

$$0 \mapsto c \mapsto c^2 + c \mapsto (c^2+c)^2 + c \mapsto \ldots$$

$$\textit{is bounded}\}$$

# The parameter plane of $z^2 + c$





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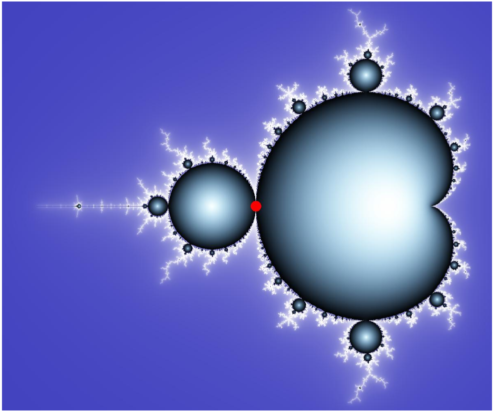

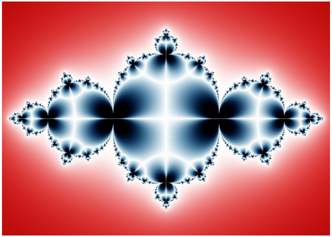

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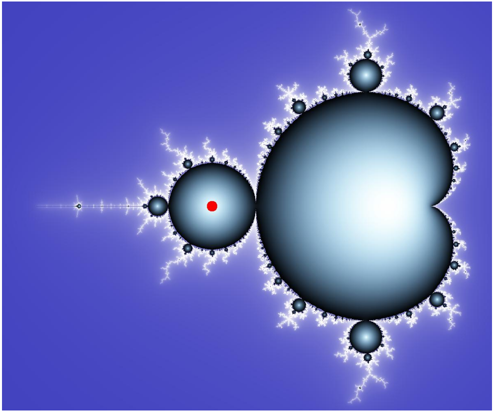

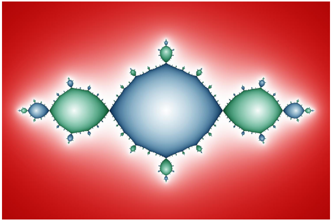

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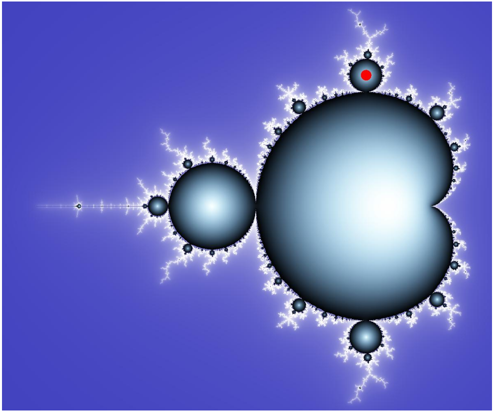

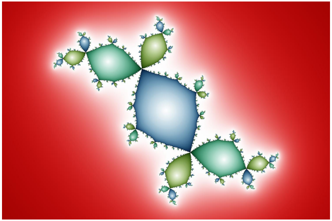

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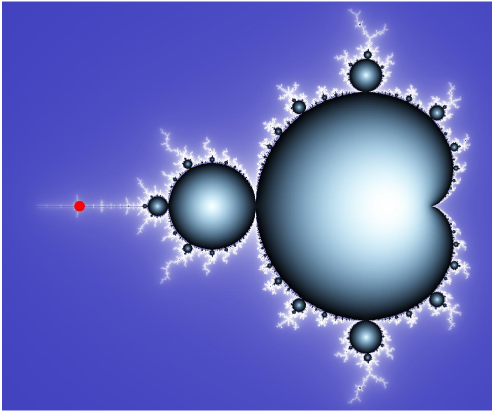

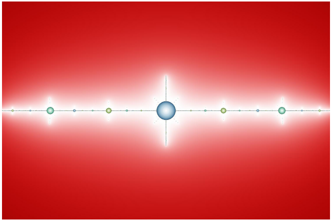

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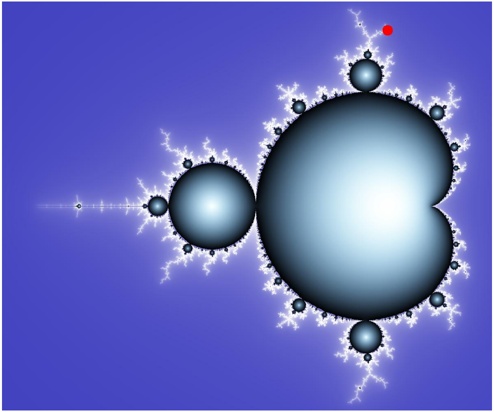

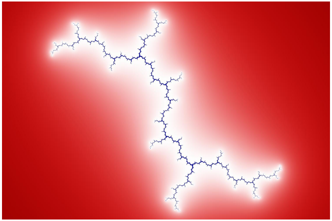

# Topological models for polynomials

Let $P$ be a polynomial of degree $d$ with connected Julia set $J(P)$ and basin of infinity $\Omega_P$.

There exists a conformal map $\phi : \mathbb{C} \setminus \overline{\mathbb{D}} \to \Omega_P$ which conjugates $\sigma_d(z) = z^d$ and $P|_{\Omega_P}$. If $J(P)$ is locally connected, then $\phi$ extends and there is a continuous map $\phi : \mathbb{S} \to J(P)$ which semi-conjugates $\sigma_d|_{\mathbb{S}}$ and $P|_{J(P)}$.

Set $x \sim_P y$ iff $\phi(x) = \phi(y)$;

then the equivalence relation $\sim_P$ on $\mathbb{S}$ is called the $\sigma_d$-invariant lamination generated by $P$.

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# Invariant laminations

## Definition (Lamination)

*An equivalence relation $\sim$ on $\mathbb{S}$ is called a lamination if:*

1. *Convex hulls of classes are disjoint,*
2. *the graph of $\sim$ is a closed subset of $\mathbb{S} \times \mathbb{S}$,*
3. *each equivalence class of $\sim$ is finite.*

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1. $\sim$ *is* forward invariant*: for a $\sim$-class $\mathfrak{g}$, the set $\sigma_d(\mathfrak{g})$ is a $\sim$-class,*
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Let $\sim$ be a $\sigma_d$-invariant lamination. The quotient space $J_\sim = \mathbb{S}/\sim$ is called a topological Julia set and the map $f_\sim : J_\sim \to J_\sim$ induced by $\sigma_d|_\mathbb{S}$ a topological polynomial.

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A complex polynomials of degree $d$, with locally connected Julia set, corresponds to a topological poynomial which is defined by a $\sigma_d$-invariant lamination $\approx$ on the unit circle $\mathbb{S}$.

To study the parameter space of all polynomials of degree $d$, with connected Julia sets, Thurston proposed studying the space of all $\sigma_d$-invariant laminations.

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Convex hulls of $\sim$-classes are pairwise disjoint.
Consider all their edges; this is a closed family of chords $\mathcal{L}_P$.

Thurston studied the dynamics of families of chords similar to $\mathcal{L}_P$ without referring to polynomials. Such families of chords are called $\sigma_d$-invariant geometric laminations (geolaminations).

To each geolamination $\mathcal{L}$ we associate the union of all its leaves which is a continuum called the solid of $\mathcal{L}$.

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# Limits of laminations

Geolaminations provide a way of putting a natural topology on the set of laminations (through the Hausdorff metric on the set solids of geolaminations).

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# Geolaminations

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*A closed family $\mathcal{L}$ of pairwise disjoint chords in $\mathbb{D}$ is called a geolamination.*

Elements of $\mathcal{L}$ are called leaves and the closures of components of $\mathbb{D} \setminus \bigcup \mathcal{L}$ gaps. For a gap or leaf $G$ we denote by $\sigma_d(G)$ the convex hull of $\sigma_d(G \cap \mathbb{S})$.

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The above definition is a slight modification of Thurston's definition. It can be shown that any geolamination which is $\sigma_d$-invariant is also invariant in the sense of Thurston (i.e., these geolaminations are also gap invariant).

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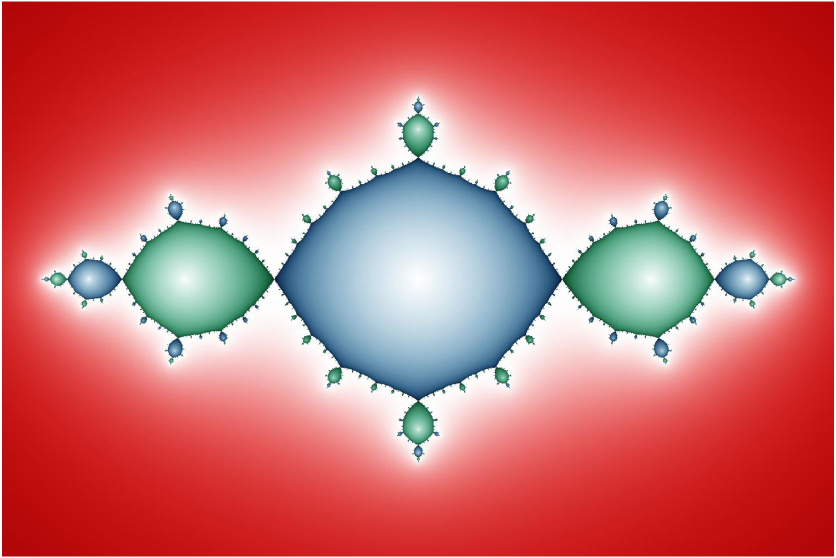
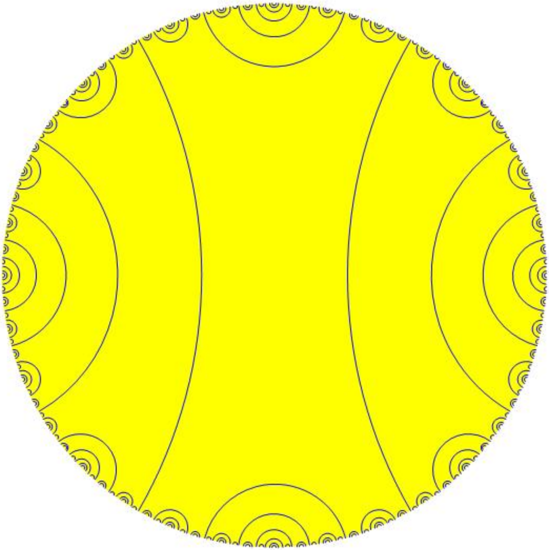
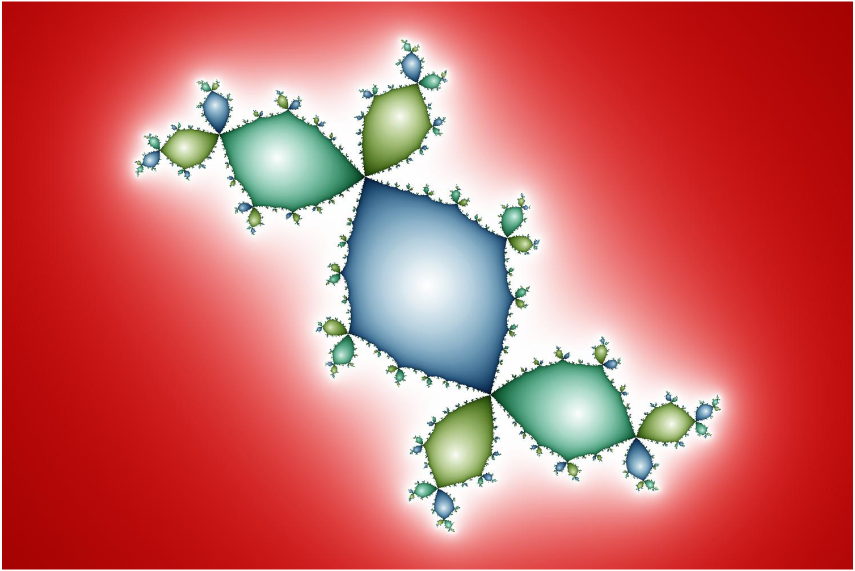
# Basilica: $f(z) = z^2 - 1$



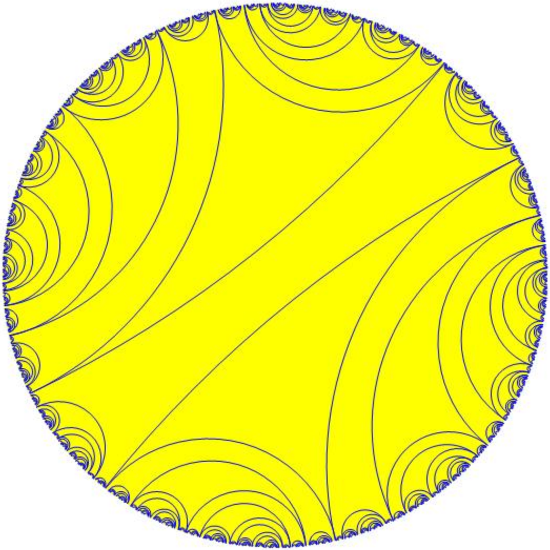
# Geolamination for $z^2 - 1$



# Rabbit: $f(z) = z^2 - 0.12.. + 0.74..i$



# Geolamination for the rabbit



# Parameterization of laminations

Recall that we want to study the space of invariant laminations; each lamination corresponds to a geolamination.

A geolamination consists of a collection of leaves.

Which leaves in such a geolamination determine the entire geolamination?

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Hence to study the space of laminations we can study the space of critical sets.

Every $\sigma_d$-invariant geolamination has at most $d-1$ critical sets.

An *ordered* collection $\mathcal{C}$ of $d-1$ critical chords so that no two intersect in $\mathbb{D}$ and their union does not contain a SCC, is called a full collection of critical chords.

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# Full collections of critical chords

The space of full collections of critical chords is a:

circle if $d = 2$,

a 2-manifold if $d = 3$,

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Distinct full collections of critical chords may well correspond to the same lamination and, hence, the same topological polynomial. When is that the case?

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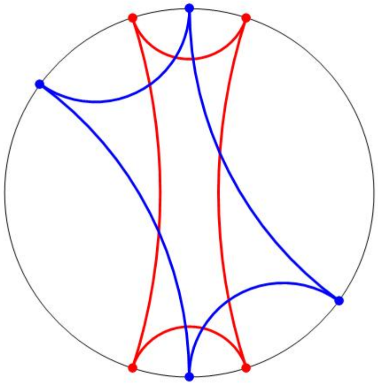
# Linkage

If two polygons (e.g., quadrilaterals) have alternating vertices, we call them strongly linked:



# $\sigma_d$-invariant laminations

Suppose that $Q$ is a quadrilateral with vertices $a_0 < a_1 < a_2 < a_3$ in $\mathbb{S}$ so that $\sigma_d(a_0) = \sigma_d(a_2)$ and $\sigma_d(a_1) = \sigma_d(a_3)$ and $\sigma_d(Q)$ is a leaf. Then diagonals of $Q$ are critical chords called spikes and $Q$ is called a critical quadrilateral.

If all critical sets of a $\sigma_d$-invariant geolamination $\mathcal{L}$ are critical quadrilaterals, then there are $d-1$ of them. Choosing one spike in each of them, we get a collection of $d-1$ critical chords called a complete sample of spikes. Call a no loop collection of $d-1$ critical chords a full collection. If $\mathcal{L}$ is a geolamination which corresponds to a lamination and all its critical sets are critical quadrilaterals, then a complete sample of spikes is a full collection.

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### Definition (Quadratic criticality)

*Let* $(\mathcal{L}, \mathrm{QCP})$ *be a geolamination with an ordered* $(d-1)$*-tuple* $\mathrm{QCP}$ *of critical quadrilaterals that are gaps or leaves of* $\mathcal{L}$ *such that any complete sample of spikes is a full collection. Then* $\mathrm{QCP}$ *is called a quadratically critical portrait (qc-portrait) for* $\mathcal{L}$ *and is denoted by* $\mathrm{QCP}$ *while the pair* $(\mathcal{L}, \mathrm{QCP})$ *is called a geolamination with a qc-portrait.*

# $\sigma_d$-invariant laminations

We assume that our geolaminations come with ordered qc-portraits. We allow for degenerate quadrilaterals:

**Definition**

*A (generalized) critical quadrilateral $Q$ is the convex hull of ordered collection of at most 4 points $a_0 \le a_1 \le a_2 \le a_3 \le a_0$ in $\mathbb{S}$ so that $\overline{a_0 a_2}$ and $\overline{a_1 a_3}$ are critical chords called spikes. Two critical quadrilaterals are viewed as equal if their marked vertices coincide up to a circular permutation of indices.*

A collapsing quadrilateral is a critical quadrilateral, whose $\sigma_d$-image is a leaf. A critical quadrilateral $Q$ has two intersecting spikes and is either a collapsing quadrilateral, a critical leaf, an all-critical triangle, or an all-critical quadrilateral.

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Lemma

*The family of all $\sigma_d$-invariant geolaminations with qc-portraits is closed.*

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*A critical cluster of $\mathcal{L}$ is by definition a convex subset of $\overline{\mathbb{D}}$, whose boundary is a union of critical leaves.*

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## Definition (Linked geolaminations)

*Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be geolaminations with qc-portraits $\mathrm{QCP}_1 = (C_1^i)_{i=1}^{d-1}$ and $\mathrm{QCP}_2 = (C_2^i)_{i=1}^{d-1}$ and a number $0 \le k \le d-1$ such that:*

1. *for every $i$ with $1 \le i \le k$, the sets $C_1^i$ and $C_2^i$ are either strongly linked critical quadrilaterals or share a spike;*
2. *for each $j > k$ the sets $C_1^j$ and $C_2^j$ are contained in a common critical cluster of $\mathcal{L}_1$ and $\mathcal{L}_2$.*

*Then we say that $\mathcal{L}_1$ and $\mathcal{L}_2$ are linked geolaminations. The critical sets $C_1^i$ and $C_2^i$, $1 \le i \le d-1$ are called associated critical sets.*

# Generic topological polynomials

**Definition (Generic topological polynomial)**

*A topological polynomial is generic if all critical sets of the corresponding geolamination are finite.*

If a topological polynomial is not generic then it either has a periodic infinite critical set with a periodic point on its boundary or an infinte non-periodic critical set which maps to a periodic infinte critical set.

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Every $\sigma_d$ invariant geolamination has at most $d-1$ critical sets.

We can associate to every generic topological polynomial a geolamination with a qc-portrait by inserting a full collection of $d-1$ generalized critical quadrilaterals into the critical sets.

Two generic topological polynomials are linked if the resulting geolaminations with qc-portraits are linked.

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**Theorem (Main Theorem)**

*If two generic topological polynomials have linked geolaminations, then the corresponding laminations and hence the two topological polynomials are the same.*

# Generic topological polynomials

If $J_\sim$ is the topological Julia set of a generic topological polynomial then every gap $G$ of the corresponding lamination is either finite or a periodic Siegel gap $U$ (so that the first return map on the boundary is semi-conjugate to an irrational rotation of a circle), or a non-periodic gap $V$ so that its boundary maps monotonically to the boundary of a periodic Siegel gap.

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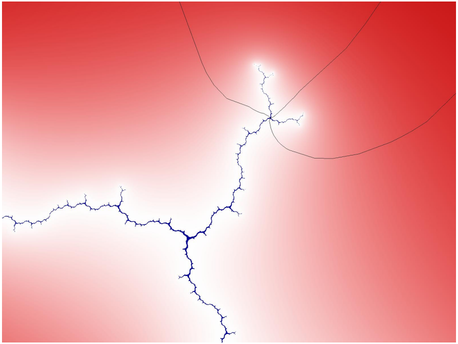
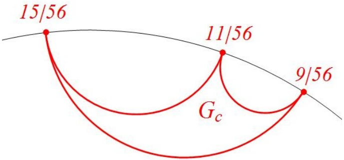
# Minor tags of generic topological polynomials

Let $P$ be a generic quadratic topological polynomial.

Then the associated geolamination has a unique critical gap/leaf $G_P$. Then $\sigma_2(G_P)$ is a gap, leaf or point in $\mathbb{D}$ which is called the minor tag of $P$.





# Minor tags of generic quadratic topological polynomials

The following theorem follows from classical results of Douady, Hubbard and Thurston.

### Theorem (Thurston)

*If $P_1$ and $P_2$ are two distinct generic quadtratic polynomials, Then their minor tags are disjoint and this collection of all minor tags is upper-semicontinuous. Hence the closure of the collection of all such minor tags is a lamination: the space of quadratic generic topological polynomials is a lamination itself!*

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*There exists a continuous function from the space of dendritic polynomials $\mathcal{MD}_2$ to the quotient space which identifies each minor tag to a point.*

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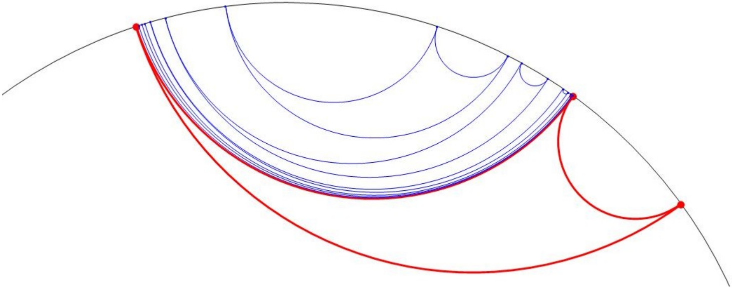
# Upper semicontinuity

**Definition**

We say that a family of sets $G_\alpha$ is upper-semicontinuous if whenever $x_n \in G_{\alpha_n}$ converges to $x_\infty \in G_\alpha$, then $\limsup G_{\alpha_n} \subset G_\alpha$.



# Majors and minors for $d = 2$

Thurston defines for each $\sigma_2$-invariant geolamination $\mathcal{L}$ as its major a leaf $M_{\mathcal{L}} \in \mathcal{L}$ of maximal length and as its minor $m_{\mathcal{L}} = \sigma_2(M_{\mathcal{L}})$.

For each dendritic polynomial $P_c = z^2 + c$ of degree 2 with associated lamination $\sim_P$ and geolamination $\mathcal{L}_P$, $m_{\mathcal{L}_P} \subset \sigma_2(G_c)$ is an edge of $\sigma_2(G_c)$. (In fact, the shortest edge of $\sigma_2(G_c)$.)

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*The collection of all minors of all $\sigma_2$-invariant geolaminations is itself a geolamination, called QML=$\{m_{\mathcal{L}}\}$ (for Quadratic Minor Lamination). Moreover $\mathcal{M}_2^{Comb} = \mathbb{S}/\mathrm{QML}$ is a locally connected continuum.*

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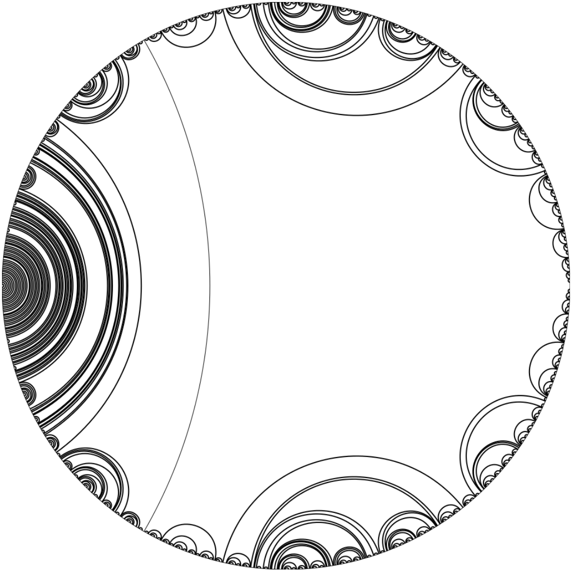
# Quadratic Minor Lamination

The quadratic minor lamination $\mathrm{QML} = \{m_{\mathcal{L}}\}$ contains all minor tags of dendritic quadratic polynomials and their limits.

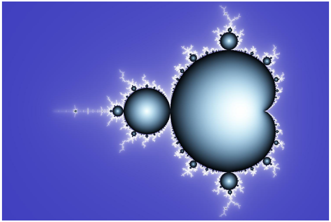
# Quadratic Minor Lamination

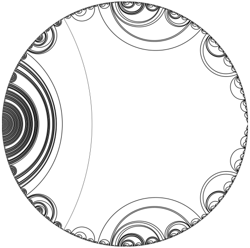


# The MLC conjecture

## Conjecture

The boundary of the Mandelbrot set $\mathcal{M}_2$ is homeomorphic to the quotient space $\mathcal{M}_2^{Comb} = \mathbb{S}/\mathrm{QML}$. In other words, the Mandelbrot set is obtained from $\overline{\mathbb{D}}$ by collapsing all leaves of $\mathrm{QML}$.





# Tagging cubic polynomials

Quadratic polynomials $P_c = z^2 + c$ are tagged by their critical value. Generic Quadratic Laminations are tagged by minors tags which are the image of the critical set. Quadratic geolaminations are tagged by minors (images of major leaves in the boundary of the critical set).

In case $d = 2$ one can think of a major as a leaf whose length is closest to $\frac{1}{2}$. Similarly, when $d = 3$, one could find two appropriate leaves whose length is closest to $\frac{1}{3}$ and declare their images to be the two minors of $\mathcal{L}$.

However, unlike the case $d = 2$, minors do not uniquely determine the majors and two distinct laminations may well have the same pair of minors.

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In case $d = 2$ one can think of a major as a leaf whose length is closest to $\frac{1}{2}$. Similarly, when $d = 3$, one could find two appropriate leaves whose length is closest to $\frac{1}{3}$ and declare their images to be the two minors of $\mathcal{L}$.

However, unlike the case $d = 2$, minors do not uniquely determine the majors and two distinct laminations may well have the same pair of minors.

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Every cubic polynomial is affinely conjugate to a polynomial of the form

$$P(z) = z^3 + 3cz^2 + v$$

which has critical points 0 and $-2c$. We consider $P = (P, -2c, 0)$ as a marked polynomial.

Note that $v = P(0)$ is a critical value called the minor tag and that $P(c) = P(-2c)$. We call $c = (-2c)^*$ the co-critical point (of $-2c$).

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*The mixed tag of the marked polynomial $(P, -2c, 0)$ is the pair*

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Let $(P, c, w)$ be a marked dendritic polynomial. Then there exists a lamination $\sim$, a quotient map $\pi : \mathbb{S} \to J_\sim = \mathbb{S}/\sim$ and a monotone map $m : J(P) \to J_\sim$, where the topological Julia set $J_\sim$ is a dendrite. Let $\mathcal{L}_\sim$ denote the associated geolamination.

For each $z \in J$, let $G_z$ denote the convex hull of $\pi^{-1} \circ m(z)$. We call $(\mathcal{L}_\sim, G_c, G_w)$ a marked geolamination. Note that $G_c$ and $G_w$ are the critical sets of $\mathcal{L}_\sim$.

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# Tagging cubic dendritic polynomials

The mixed tag of $(P, c, w)$

$$Tag(P, c, w) = (G_{c^*} \times \sigma_3(G_w)) \subset \overline{\mathbb{D}} \times \overline{\mathbb{D}}.$$

Note that each tag is the product of two sets each of which is either a point, a leaf or a gap. Hence we may think of these objects as “leaves and gaps” in a higher dimensional lamination in $\overline{\mathbb{D}} \times \overline{\mathbb{D}}$.

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# Tagging generic cubic topological polynomials $P$

Every generic topological has either one or two finite critical sets which are critical gap(s) or leaves(s) of the associated geolamination $\mathcal{L}$.

We consider *marked critical sets* so if their are two critical sets we can call one $G_c$ and the other $G_w$. Denote by $G_{c^*}$ (the co-critical set) the gap/leaf/point of $\mathcal{L}$, distinct from $G_c$, which has the same image as $G_c$.
If $\mathcal{L}$ has only one critical set put $G_c = G_{c^*} = G = G_w$.
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# An application of our results

Theorem

*Mixed tags of generic cubic topological polynomials are disjoint or coincide. The closure of the collection of all mixed tags CML of cubic generic topological polynomials is itself a (higher dimensional) "lamination" in* $\overline{\mathbb{D}} \times \overline{\mathbb{D}}$.

### Corollary

*There exists a continuous function from the space of marked dendritic cubic polynomials* $\mathcal{MD}_3$ *to the quotient space* $\mathcal{MD}_3^{Comb} = \overline{\mathbb{D}} \times \overline{\mathbb{D}}/\mathrm{CML}$ *obtained by identifying individual tags to points.*

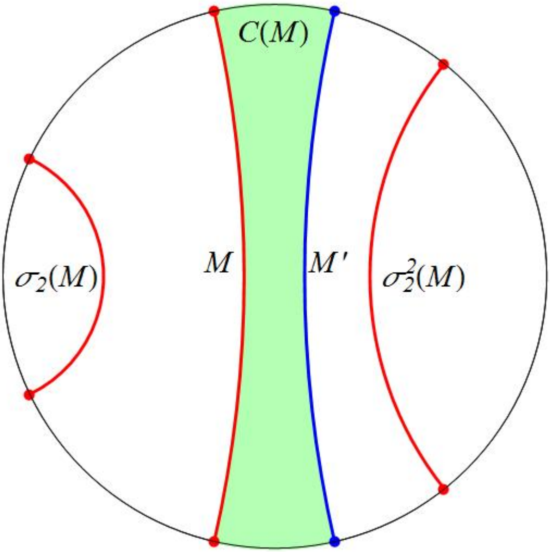
# Tags of quadratic invariant laminations

Recall that the minor $m_{\mathcal{L}}$ of a $\sigma_2$-invariant geolamination $\mathcal{L}$ is the image of a longest leaf $M$ of $\mathcal{L}$ called a major.

Thurston defined $\mathrm{QML} = \{m_{\mathcal{L}}\}$ as the set of all minors.

He proved, in particular, that distinct minors are unlinked.

# The central strip



# The Central Strip Lemma

Lemma (Thurston)

*Given two majors $M$, $M'$, let $C(M)$ be the open strip in $\mathbb{D}$ between $M$ and $M'$ (the central strip); Thurston showed that no eventual image of $M$ can enter $C(M)$.*

This implies that:

1. $\sigma_2$ cannot have wandering triangles,
2. vertices of a finite periodic gap belong to one cycle,
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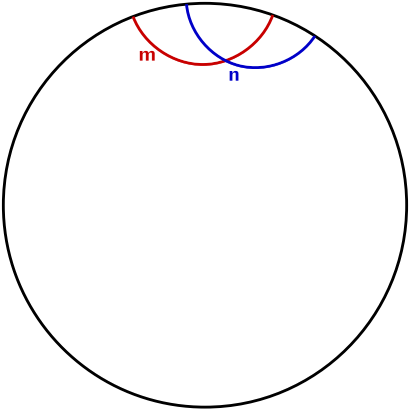
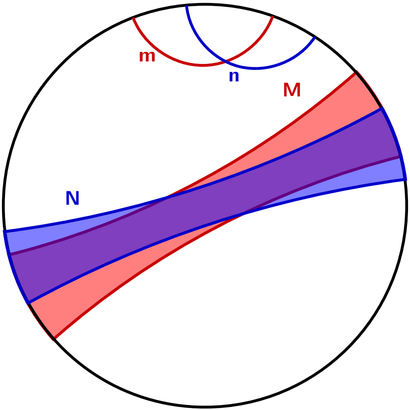
# Sketch of Unlinkage Proof

Suppose $m, n$ are linked minors.



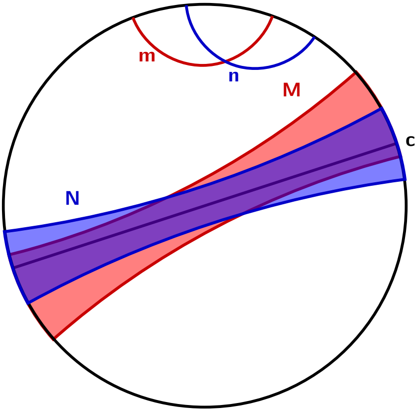
# Sketch of Unlinkage Proof

Then their majors' central strips overlap.



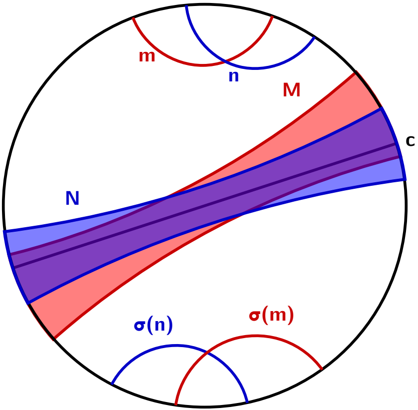
# Sketch of Unlinkage Proof

Choose $c$ a critical leaf in the intersection with preperiodic endpoints. By CSL, no image of $M, N$ can meet $c$.



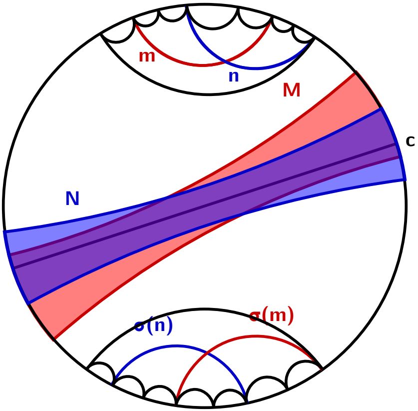
# Sketch of Unlinkage Proof

Hence $m, n$ are carried forward in order by $\sigma_2^i$.



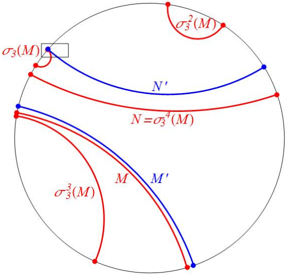
# Sketch of Unlinkage Proof

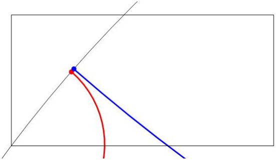
$m, n$ are contained in a finite preperiodic gap of the geolamination generated by $c$, contradicting transitivity.



# Higher degree case

The Central Strip Lemma fails for cubics;



consequences are:

1. the existence of wandering triangles for cubics,
2. the existence of periodic gaps with two cycles of vertices.

# Accordions

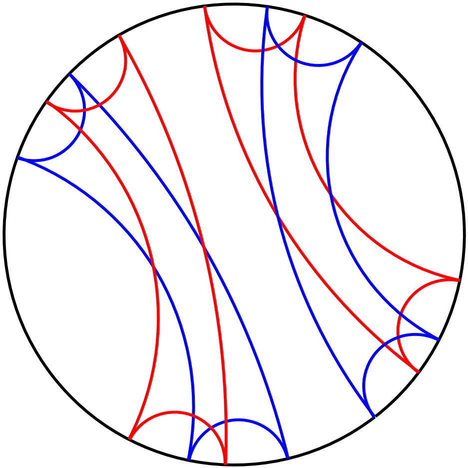
### Definition

For linked geolaminations $\mathcal{L}_1$, $\mathcal{L}_2$ with qc-portraits, an accordion is the union $\mathcal{A}$ of a leaf $\ell$ of $\mathcal{L}_1$ with the leaves of $\mathcal{L}_2$ linked with $\ell$.

Accordions resemble gaps of *one* geolamination b/c they *preserve order under $\sigma_d$*.

# Accordions

If $\ell_1 \in \mathcal{L}_1$, then each critical set of $\mathcal{L}_2$ has a spike unlinked with $\ell_1$.



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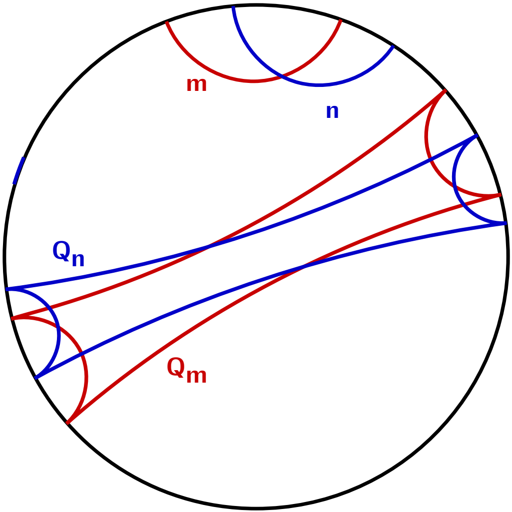

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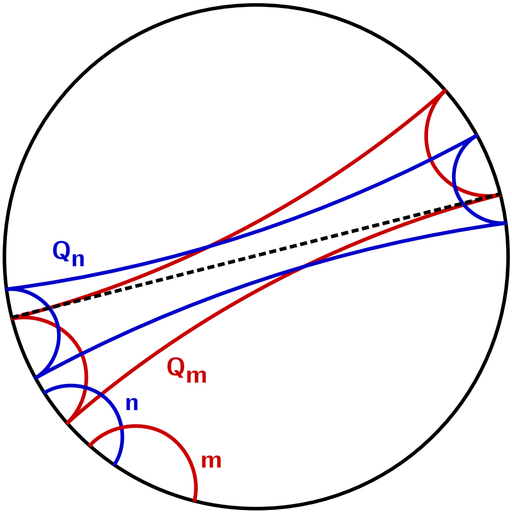

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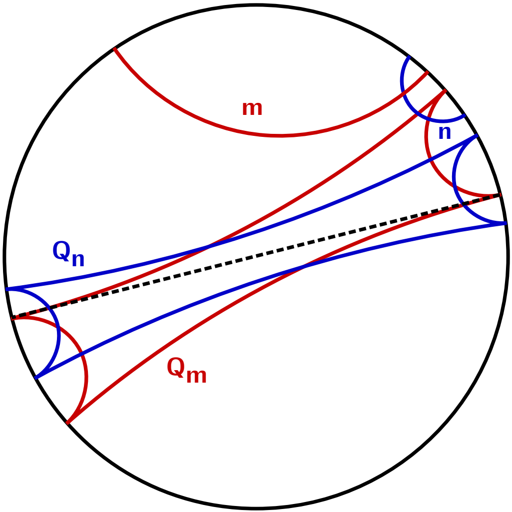

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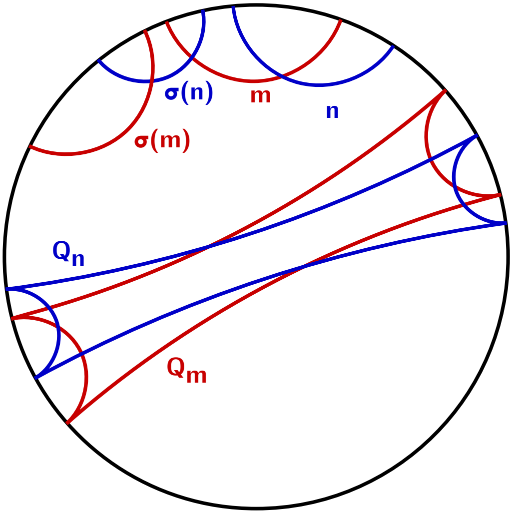

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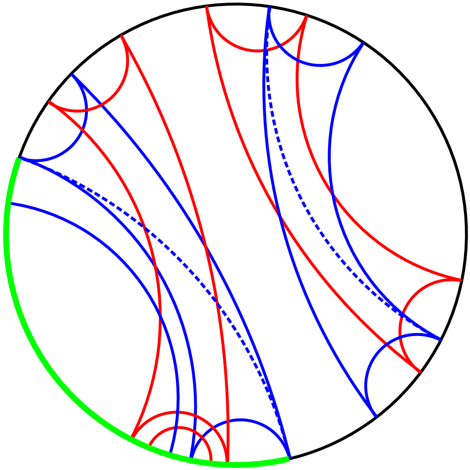

### Definition

*Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be geolaminations. Suppose that there are geolaminations with qc-portraits $(\mathcal{L}_1^m, \mathrm{QCP}_1)$, $(\mathcal{L}_2^m, \mathrm{QCP}_2)$ such that $\mathcal{L}_1 \subset \mathcal{L}_1^m$, $\mathcal{L}_2 \subset \mathcal{L}_2^m$. Then we say that $\mathcal{L}_1$ and $\mathcal{L}_2$ are linked if $(\mathcal{L}_1^m, \mathrm{QCP}_1)$ and $(\mathcal{L}_2^m, \mathrm{QCP}_2)$ are linked.*

**Theorem**

*Suppose that $\mathcal{L}_a$ and $\mathcal{L}_b$ are linked laminations. Consider linked chords $\ell_a$, $\ell_x$ and the set $B = \mathrm{CH}(\ell_a, \ell_x)$. Suppose that for all $n$, $|\sigma^n(B) \cap \mathbb{S}| = 4$ and $B$ is not wandering. Then*

(a) *the set $B$ is a stand-alone gap of some preperiod $r \geq 0$;*

(b) *if $X$ is the union of polygons in the forward orbit of $\sigma_d^r(B)$ and $Q$ is a component of $X$, then $Q \cap \mathbb{S}$ is a finite set, the vertices of $\sigma_d^r(B)$ are periodic of the same period and belong to two, or three, or four distinct periodic orbits, and the first return map on $Q \cap \mathbb{S}$ can be identity only if $Q = \sigma_d^r(B)$ is a quadrilateral,*

(c) *the leaves $\ell_a$, $\ell_x$ are (pre)periodic of the same eventual period of endpoints.*