Projective Fraïssé limits and homogeneity for tuples of points of the pseudoarc

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Outline of Topics

The pseudoarc and projective Fraïssé limits

Direct Fraïssé limits

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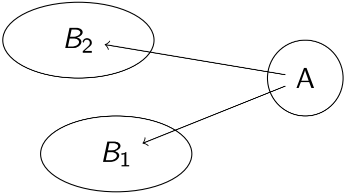

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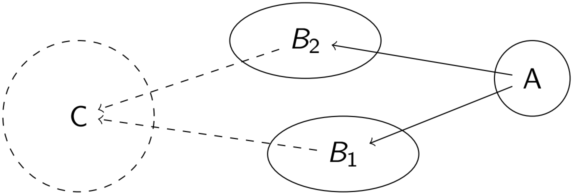

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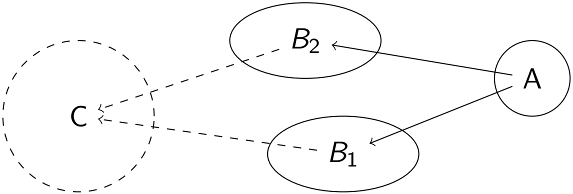

Fraïssé: Countable Fraïssé families have unique **countable limit structures**.

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The metric completion $\mathbb{U}$ of $\mathbb{U}_0$ is the **Urysohn space**, the unique universal separable, complete metric space that is ultrahomogeneous with respect to finite subspaces.

Aim: By analogy with the above approach, develop a logic/combinatorics-based point of view to:

— find **canonical/combinatorial models** for some **topological spaces**, for example, the pseudoarc, the Menger compacta, the Brouwer curve etc.;

— find a unified approach to topological **homogeneity** results for these spaces.

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the pseudoarc $P =$ a certain compact, connected, second countable space

the pre-pseudoarc $\mathbb{P} =$ the Cantor set and a certain compact equivalence relation R on it with $\mathbb{P}/R = P$ and with a certain relationship to a family of finite structures

the augmented pre-pseudoarc $\mathbb{P}_{RU} = \mathbb{P}$ with additional structure

The pseudoarc

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This P is called the **pseudoarc**.

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The pseudoarc is a **hereditarily indecomposable** continuum, that is, if $C_1, C_2 \subseteq P$ are continua with $C_1 \cap C_2 \neq \emptyset$, then $C_1 \subseteq C_2$ or $C_2 \subseteq C_1$.

Projective Fraïssé limits

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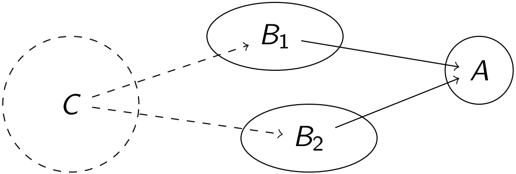
$\mathcal{F}$ is called a **projective Fraïssé family** if it has **Joint Epimorphism Property** and **Projective Amalgamation Property**.



M is a **topological structure for** $\mathcal{F}$ if

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— M is a compact, 0-dimensional, second countable space,

— each relation symbol is interpreted as a closed relation on M,

— each continuous function $M \to X$, with X finite, factors through an epimorphism $M \to A$ for some $A \in \mathcal{F}$.

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- for each $A \in \mathcal{F}$ there is an epimorphism $\mathbb{F} \to A$ (**projective universality**) and
- for each $A \in \mathcal{F}$ and epimorphisms $f : \mathbb{F} \to A$ and $g : \mathbb{F} \to A$, there is an automorphism $\phi \colon \mathbb{F} \to \mathbb{F}$ with $f \circ \phi = g$ (**projective ultrahomogeneity**).

Connection with the pseudoarc

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Irwin–S.: The family of finite R-structures is a projective Fraïssé family.

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Irwin–S.: $\mathbb{P}/R^{\mathbb{P}}$ is the pseudoarc.

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Bing: The pseudoarc is homogeneous, that is, for any $x, y \in P$, there exists $f \in \mathrm{Homeo}(P)$ such that $f(x) = y$.

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Appropriate homogeneity for tuples holds as well.

Partial homogeneity of the pre-pseudoarc

Partial homogeneity of $\mathbb{P}$

Types

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A set $K \subseteq \mathbb{P}$ is called an R-**substructure** if it is compact, non-empty, and for each finite R-structure A and each epi $f : \mathbb{P} \to A$, $f[K]$ is an interval.

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Note that

$$\mathrm{tp}^p \subsetneq \mathrm{Tp}^p.$$

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$$\mathrm{tp}^{\mathrm{p}}(f) = \{f[K]\colon K \in \mathrm{tp}^{p}\} \quad \text{and} \quad \mathrm{Tp}^{\mathrm{p}}(f) = \{f[K]\colon K \in \mathrm{Tp}^{p}\}.$$

Minimal types and independence

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$p \in \mathbb{P}$ **has minimal types** if for each continuous $f\colon \mathbb{P} \to X$ with X finite

$$\mathrm{tp}^{\mathrm{p}}(f) = \mathrm{Tp}^{\mathrm{p}}(f).$$

$p, q \in \mathbb{P}$ are **independent** if p and q do not both belong to a *proper* R-substructure of $\mathbb{P}$.

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A tuple of points is called **independent** if every two of its elements are.

Main theorem for partial homogeneity of $\mathbb{P}$

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Theorem (S.-Tsankov)

Let $p_1, \ldots, p_n \in \mathbb{P}$ be independent and p_i have minimal types, for each i, and
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Theorem (S.-Tsankov)

Let $p_1, \ldots, p_n \in \mathbb{P}$ be independent and p_i have minimal types, for each i, and
let $q_1, \ldots, q_n \in \mathbb{P}$ be independent and q_i have minimal types, for each i,
then there exists an automorphism $\phi\colon \mathbb{P} \to \mathbb{P}$ such that $\phi(p_i) = q_i$.

Augmented R-structures as a projective Fraïssé family

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If U is a chain on X and $f\colon X \to Y$ is a surjection, then

$$f(U) = \{f[I]\colon I \in U\}$$

is also a chain.

Side-observation

$p \in \mathbb{P}$ has minimal types if and only if, for each continuous $f : \mathbb{P} \to X$ with X finite, $\mathrm{Tp}^p(f)$ is a chain.

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(i) (A, R^A) is an R-structure;

(ii) U_i^A is a chain of intervals in A, for all $1 \leq i \leq n$.

Let A and B be RU-structures. Then $f\colon B \to A$ is an RU-**epimorphism** if it is an R-epimorphism and

$$f(U_i^B) = U_i^A \text{ for each } i.$$

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Theorem (S.–Tsankov)

The family of finite RU-structures with RU-epimorphisms forms a projective Fraïssé family.

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The **proof** uses a combinatorial **chessboard theorem** due to Steinhaus.

Generic tuples and their characterization

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The tuple $(p_1^{RU}, \ldots, p_n^{RU})$ is called **generic**.

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Theorem (S.–Tsankov)

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Side-observation

$$U_i^{\mathbb{P}_{RU}} = \mathrm{Tp}^{p_i}.$$

Transfer theorem and homogeneity of the pseudoarc

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(i) *$x_i = p_i/R^{\mathbb{P}}$ for some $p_i \in \mathbb{P}$ with $(p_1, \dots, p_n)$ independent and each p_i having minimal types, for $1 \leq i \leq n$;*

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(ii) $\phi(x_i) = y_i$*, for* $1 \leq i \leq n$*.*

The **proof** is purely combinatorial.

Corollary (Bing)

Let $y_1, \ldots, y_n \in P$ *be in general position, and let* $z_1, \ldots, z_n \in P$ *be in general position.*

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Let $y_1, \ldots, y_n \in P$ be in general position, and let $z_1, \ldots, z_n \in P$ be in general position. There exists a homeomorphism of P mapping y_i to z_i for each $1 \leq i \leq n$.

The Menger curve μ_1