Definable Versions of Menger's Conjecture

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Definition 1

A space is **Menger** if whenever $\{\mathcal{U}_n\}_{n<\omega}$ is a sequence of open covers, there exist finite $\{\mathcal{V}_n\}_{n<\omega}$ such that $\mathcal{V}_n \subseteq \mathcal{U}_n$ and $\bigcup\{\mathcal{V}_n : n < \omega\}$ is a cover.

Menger: Are Menger subsets of $\mathbb{R}$ σ-compact?

Proposition 1 (Hurewicz 1925)

Completely metrizable (indeed, analytic) Menger spaces are σ-compact.

Example 1 (Chaber-Pol 2002, Tsaban-Zdomskyy 2008)

There are Menger subsets of $\mathbb{R}$ which are not σ-compact.

Problem 1

Are "definable" Menger subsets of $\mathbb{R}$ σ-compact?

Proposition 2 (Miller-Fremlin 1988)

$V = L$ implies there is a CA (complement of analytic) Menger subset of $\mathbb{R}$ which is not σ-compact.

Definition 2

The **projective** subsets of $\mathbb{R}$ are obtained by closing the Borel sets under continuous image and complementation.

Definition 3

Let $X \subseteq {}^{\omega}\omega$. In the game $G(X)$, player *I* picks $a_0 \in \omega$, player II picks $a_1 \in \omega$, player I picks $a_2 \in \omega$, etc. I wins iff $\{a_n\}_{n<\omega} \in X$. $G(X)$ is **determined** if either I or II has a **winning strategy**. The **Axiom of Projective (co-analytic) Determinacy** says *all projective (co-analytic)* games are determined.

Theorem 3 (Miller-Fremlin, TT)

PD (CD) implies all projective (co-analytic) Menger subsets of $\mathbb{R}$ are σ-compact.

Theorem 4 (TT)

If there is a measurable cardinal, then co-analytic Menger subsets of $\mathbb{R}$ are σ-compact.

It is known that CD is equiconsistent with a measurable.

Problem 2

Without large cardinals, is it consistent that co-analytic (projective?) Menger subsets of $\mathbb{R}$ are σ-compact?

Theorem 5 (T-Todorcevic-T)

If it is consistent there is an inaccessible cardinal, it is consistent that projective Menger subsets of $\mathbb{R}$ are σ-compact.

Proof.

Use a strengthening **OCA(projective)** of Todorcevic's **Open Coloring Axiom** mentioned in Feng (1993):

OCA(projective) If $X \subseteq \mathbb{R}$ is uncountable projective and $[X]^2 = K_0 \cup K_1$ is a partition with K_0 open in the relative topology, then either there is a **perfect** $A \subseteq X$ with $[A]^2 \subseteq K_0$, or $X = \bigcup_{n<\omega} A_n$, with $[A_n]^2 \subseteq K_1$ for all $n < \omega$.

Theorem 6 (Feng)

OCA(projective) is equiconsistent with an inaccessible cardinal.

Hurewicz Dichotomy for projective sets Let E be a compact metrizable space and let A, B be disjoint projective subsets of E. Either there is a σ-compact $C \subseteq E$ such that $A \subseteq C$ and $C \cap B = \emptyset$, or there is a copy F of the Cantor set such that $F \subseteq A \cup B$ and $F \cap B$ is countable dense in F. $\square$

Problem 3

Can Hurewicz' theorem be extended to non-metrizable spaces?

Definition 4

A space is **analytic** if it is a continuous image of the space $\mathbb{P}$ of irrationals.

Proposition 7 (Arhangel'skiĭ 1986)

Analytic Menger spaces are σ-compact.

Theorem 8 (TT)

Čech-complete Menger spaces are σ-compact.

Proof.

A Čech-complete Lindelöf space is a perfect pre-image of a separable metrizable space. A perfect image of a Čech-complete space is Čech-complete. A continuous image of a Menger space is Menger. A perfect pre-image of a σ-compact space is σ-compact. $\square$

Definition 5

A space is **co-analytic** if its Čech-Stone remainder is analytic.

Problem 4

Is it consistent that co-analytic Menger spaces are σ-compact?

Example 2

There is a continuous image of a co-analytic space which is not σ-compact.

Okunev's space Take the Alexandrov duplicate of $\mathbb{P}$ and collapse the non-discrete copy of $\mathbb{P}$ to a point. See Burton-Tall 2012 for details.

Theorem 9 (Tall 2016, Tokgöz 2016)

It is undecidable whether co-analytic Menger topological groups are σ-compact.

Theorem 10 (TT)

CD implies co-analytic Menger spaces are σ-compact if they either have closed sets G_δ or are $\sum$-spaces.

Productive Lindelöfness

Definition 6
A space X is **productively Lindelöf** if for every Lindelöf Y, $X \times Y$ is Lindelöf.

Proposition 11
Productively Lindelöf spaces are consistently Menger. (Repovs-Zdomskyy 2012, Alas-Aurichi-Junqueira-Tall 2011, Tall 2013)

Problem 5
Are productively Lindelöf co-analytic spaces σ-compact?

Theorem 12

CH implies productively Lindelöf, co-analytic, nowhere locally compact spaces are σ-compact.

Theorem 13

There is a Michael space (i.e. a Lindelöf space X such that $X \times \mathbb{P}$ is not Lindelöf) iff productively Lindelöf Čech-complete spaces are σ-compact.

Another generalization of definability

Definition 7 (Frolík)

A space is **K-analytic** if it is a continuous image of a Lindelöf Čech-complete space.

Example 3

Okunev's space is a K-analytic productively Lindelöf Menger space which is not σ-compact.

Theorem 14

K-analytic co-analytic Menger spaces are σ-compact.

Proof.

Such a space X is a Lindelöf p-space since both it and its remainder are Lindelöf $\sum$. Let X map perfectly onto a metrizable M. Then M is analytic and Menger, so is σ-compact, so X is also. $\square$

Definition 8

A space is **Hurewicz** if every Čech-complete space including it includes a σ-compact subspace including it.

Lemma 15

σ-compact $\rightarrow$ Hurewicz $\rightarrow$ Menger. No arrow reverses, even for subsets of $\mathbb{R}$.

Okunev's space is Hurewicz.

Definition 9 (Arhangel'skiĭ 2000)

A space is **projectively σ-compact** if each continuous separable metrizable image of it is σ-compact.

Theorem 16

Every K-analytic Menger space is Hurewicz.

Proof.

Each such space is projectively σ-compact. $\square$

Definition 10 (Rogers-Jayne 1980)

A space is **K-Lusin** if it is an injective continuous image of $\mathbb{P}$.

Problem 6

Is every Menger K-Lusin space σ-compact?

Lemma 17 (Rogers-Jayne 1980)

The following are equivalent for a K-Lusin X:

(a) *X includes a compact perfect set;*

(b) *X admits a continuous real-valued function with uncountable range;*

(c) *X is not the countable union of compact subspaces which include no perfect subsets. In particular, if X is not σ-compact, it includes a compact perfect set.*

From this, we can conclude that Okunev's space is not K-Lusin, since it is not σ-compact but doesn't include a compact perfect set.

Indeed we have:

Definition 11

A space is **Rothberger** if whenever $\{\mathcal{U}_n\}_{n<\omega}$ are open covers, there exists a cover $\{U_n\}_{n<\omega}, U_n \in \mathcal{U}_n$.

Thus *Rothberger* is a strengthening of *Menger*.

Lemma 18 (Aurichi 2010)

Rothberger spaces do not include a compact perfect set.

Theorem 19

K-analytic Rothberger spaces are projectively countable.

Proof.

They are projectively σ-compact. □

Corollary 20

K-Lusin Rothberger spaces are σ-compact.

Proof.

This follows from Lemma 17. □

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