COMPLETENESS TYPE PROPERTIES AND SPACES OF CONTINUOUS FUNCTIONS

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Initial conditions

- In this talk *space* will mean *Tychonoff space with more than one point*.
- For every space of the form $C_p(X, Y)$ considered in this talk, the spaces X and Y are such that $C_p(X, Y)$ is dense in Y^X.

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Introduction

- Pseudocompact and Baire spaces are outstanding classes of spaces, but these properties are not productive.

- Efforts have been made to define classes of spaces which contain all pseudocompact spaces, satisfy the Baire Category Theorem and are closed under arbitrary topological products.

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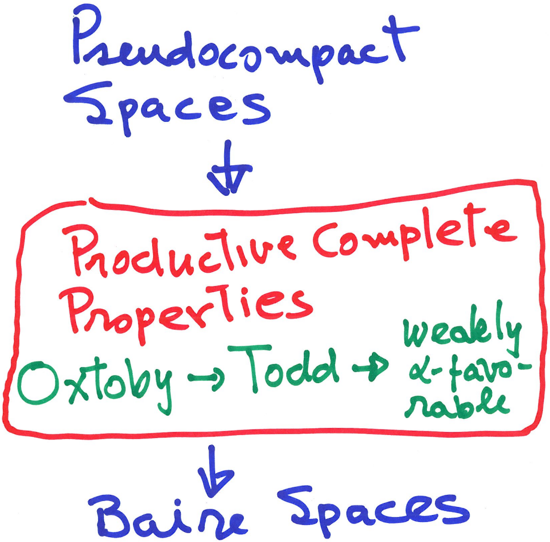
Pseudocompact Spaces
Productive Complete Properties
Oxtoby → Todd → weakly α-favorable
Baire Spaces

- One of these properties is Oxtoby completeness.

- Another is Todd completeness.

Introduction

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- Another is Todd completeness.

Definition 2.1.

A family $\mathcal{B}$ of sets in a topological space X is called π-*base*

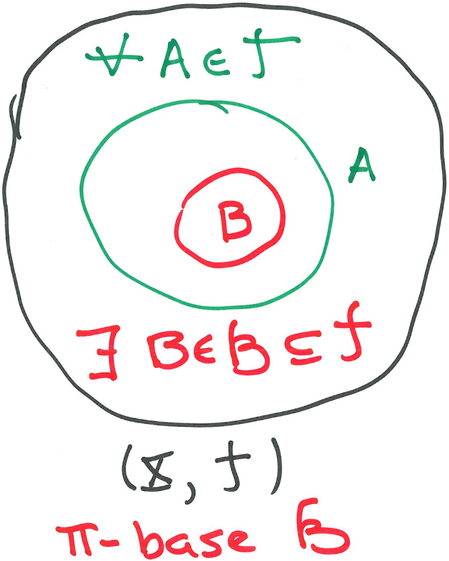
(respectively, π-*pseudobase*)

if every element of $\mathcal{B}$ is open

(respectively, has a nonempty interior)

and every nonempty open set in X contains an element of $\mathcal{B}$.

Completeness type properties



Definition 2.2.

A space X is *Oxtoby complete* (respectively, *Todd complete*) if there is a sequence

$$\{\mathcal{B}_n : n < \omega\}$$

of π-bases, (respectively, π-pseudobases) in X such that for any sequence $\{U_n : n < \omega\}$ where

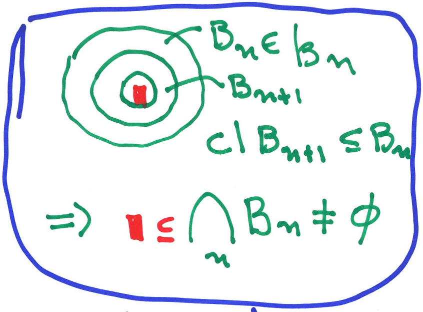
$$U_n \in \mathcal{B}_n \text{ and } \mathrm{cl}_X U_{n+1} \subseteq \mathrm{int}_X U_n \text{ for all } n,$$

then,

$$\bigcap_{n<\omega} U_n \neq \emptyset.$$

Oxtoby Sequence

$\{\mathcal{B}_n : n<\omega\}$



(X, τ)

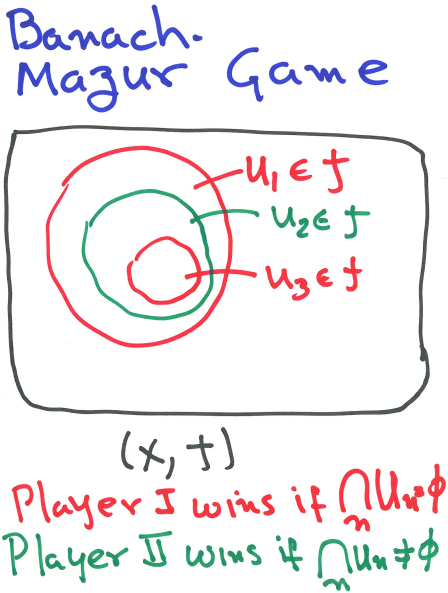
$\mathcal{B}_n$ is a π-base

There are also properties of type completeness defined by topological games:

Definition 2.3.

A space Z is *weakly α-favorable* if Player II has a winning strategy in the Banach-Mazur game $\mathrm{BM}(Z)$.

Completeness type properties



The relations between all these properties are:

Pseudocompact $\Rightarrow$ Oxtoby complete $\Rightarrow$ Todd complete

Todd complete $\Rightarrow$ weakly α-favorable $\Rightarrow$ Baire

Spaces of continuous functions

We want to say something about the completeness properties just presented but in spaces of continuous real-valued functions with the pointwise convergence topology $C_p(X)$.

Mainly, we want to relate these properties in $C_p(X)$ with topological properties defined in X.

We have for instance that:

Proposition 3.1.

$C_p(X)$ is never pseudocompact.

Proposition 3.2, van Douwen, Pytkeev

$C_p(X)$ is a Baire space iff every pairwise disjoint sequence of finite subsets of X has a strongly discrete subsequence.

Spaces of continuous functions

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Spaces of continuous functions

- Next we present the key property in X which allows us to relate the completeness type properties in $C_p(X)$:

Definition 3.3.

A space X is *u-discrete* if every countable subset of X is discrete and C-embedded in X.

- For example, every P-space is u-discrete.

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D.J. Lutzer and R.A. McCoy analyzed Oxtoby pseudocompleteness in $C_p(X)$. They proved:

Theorem 3.4, 1980

Let X be a pseudonormal space. Then, the following are equivalent:
1.- X is u-discrete.

2.- $C_p(X)$ is Oxtoby complete.

3.- $C_p(X)$ is weakly α-favorable.

4.- $C_p(X)$ is G_δ-dense in $\mathbb{R}^X$.

Afterwards, A. Dorantes-Aldama, R. Rojas-Hernández and Á. Tamariz-Mascarúa improved the Lutzer and McCoy result:

Theorem 3.5, 2015

Let X be a space with property D of van Douwen. Then, the following are equivalent:

1.-X is u-discrete.

2.- $C_p(X)$ is Todd complete.

3.- $C_p(X)$ is Oxtoby complete.

4.- $C_p(X)$ is weakly α-favorable.

5.- $C_p(X)$ is G_δ-dense in $\mathbb{R}^X$.

And A. Dorantes-Aldama and D. Shakhmatov proved:

Theorem 3.6, 2016

The following statements are equivalent:

1.- X is u-discrete.

2.- $C_p(X)$ is Todd complete.

3.- $C_p(X)$ is Oxtoby complete.

4.- $C_p(X)$ is G_δ-dense in $\mathbb{R}^X$.

Finally, S. García-Ferreira, R. Rojas-Hernández and Á. Tamariz-Mascarúa proved:

Theorem 3.7, 2016

The following conditions are equivalent.

1.- X is u-discrete;

2.- $C_p(X)$ is Todd complete;

3.- $C_p(X)$ is Oxtoby complete;

4.- $C_p(X)$ is weakly α-favorable;

5.- $C_p(X)$ is G_δ-dense in $\mathbb{R}^X$.

Weakly pseudocompact spaces

Another completeness type property which motivated the present work is the so called *weak pseudocompactness* in $C_p(X)$.

Theorem 4.1. (Hewitt, 1948)

A space X is pseudocompact if and only if it is G_δ-dense in βX (iff it is G_δ-dense in any of its compactifications).

- So, a natural generalization of pseudocompactness is:

Definition 4.2. (García-Ferreira and García-Máynez, 1994)

A space is *weakly pseudocompact* if it is G_δ-dense in some of its compactifications.

- Then, every pseudocompact space is weakly pseudocompact.

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Weakly pseudocompact spaces

Theorem 4.3. (García-Ferreira and García-Máynez, 1994)

- Every weakly pseudocompact space is Baire.
- Weak pseudocompactness is productive.

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Weakly pseudocompact spaces

- Examples of weakly pseudocompact spaces:

  1.- The non-countable discrete spaces.

  2.- (F.W. Eckertson, 1996)
  The metrizable hedgehog $J(\kappa)$ with $\kappa > \omega$.

Lemma 4.4, (Sánchez-Texis/Okunev, 2013)

Every weakly pseudocompact space is Todd complete.

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Weakly pseudocompact spaces

Can we add “$C_p(X)$ is weakly pseudocompact"to the list of the following already mentioned theorem?

Theorem

The following conditions are equivalent.

1.- X is u-discrete;

2.- $C_p(X)$ is Todd complete;

3.- $C_p(X)$ is Oxtoby complete;

4.- $C_p(X)$ is weakly α-favorable;

5.- $C_p(X)$ is G_δ-dense in $\mathbb{R}^X$.

A more general question is:

Problem 4.5.

Is there a space X for which $C_p(X)$ is weakly pseudocompact?

Main result

Regarding this problem F. Hernández-Hernández, R. Rojas-Hernández, Á. Tamariz-Mascarúa obtained the following:

Theorem 5.1, 2016

$C_p(X, G)$ is never weakly pseudocompact when G is a metrizable, separable, locally compact non compact topological group.

Main result

As corollaries we obtain

Theorem 5.2.

The space $C_p(X)$ is never weakly pseudocompact.

Theorem 5.3.

Let X be a zero-dimensional space. Then, $C_p(X, \mathbb{Z})$ is never weakly pseudocompact.

Corollary 5.4.

The spaces $\mathbb{R}^\kappa$, $\mathbb{Z}^\kappa$, $\Sigma\mathbb{R}^\kappa$ and $\Sigma\mathbb{Z}^\kappa$ are not weakly pseudocompact for every κ.

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One interesting consequence that we obtained of the results mentioned in this talk are generalizations of the classic Tkachuk Theorem:

Theorem 6.1, V. Tkachuk, 1987

$C_p(X) \cong \mathbb{R}^\kappa$ if and only if X is discrete of cardinality κ.

We obtained:

Theorem 6.2.

Let G be a separable completely metrizable topological group and X a set. If H is a dense subgroup of G^X and H is homeomorphic to G^Y for some set Y, then $H = G^X$.

Corollary 6.3

Let X be a space and let G be a separable completely metrizable topological group. If $C_p(X, G)$ is homeomorphic to G^Y for some set Y, then $C_p(X, G) = G^X$.

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