The Bolzano property and the cube-like complexes

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Introduction

Theorem (Bolzano 1817)

If a continuous $f : [a, b] \to R$ *and*

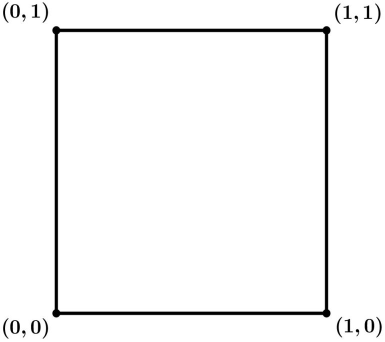
$$f(a) \cdot f(b) \leq 0,$$

then there is $c \in [a, b]$ *such that* $f(c) = 0$.

Introduction

Let $I^n = [0,1]^n$ be an n-dimensional cube in $\mathbf{R}^n$.
Its *i-th opposite faces* are defined as follows:

$$I_i^- := \{x \in I^n : \; x(i) = 0\}, \; I_i^+ := \{x \in I^n : \; x(i) = 1\}$$



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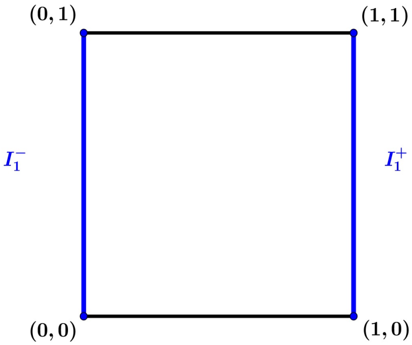

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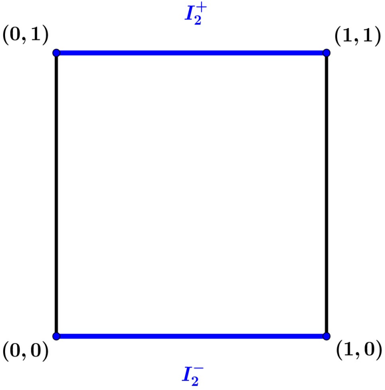

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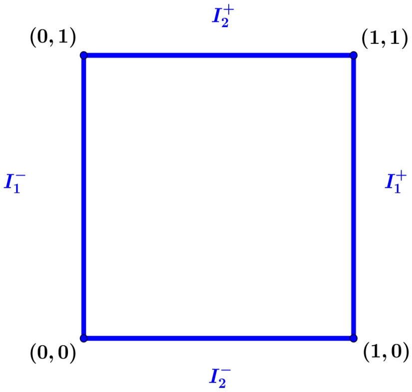

The Poincaré-Miranda theorem

Theorem (Poincaré 1883)

If a continuous

$$f = (f_1, f_2, \ldots, f_n) : I^n \to \mathbb{R}^n,$$

$$f_i(I_i^-) \subset (-\infty, 0], \qquad f_i(I_i^+) \subset [0, \infty),$$

then there is $c \in I^n$ *such that* $f(c) = (0, 0, \ldots, 0)$.

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Theorem (Miranda 1940)

The Poincaré theorem is equivalent to the Brouwer fixed point theorem.

The n-dimensional Bolzano property

Definition (Kulpa 1994)

The topological space X has *the n-dimensional Bolzano property* if there exists a family $\{(A_i, B_i) : i = 1, \ldots, n\}$ of pairs of non-empty disjoint closed subsets such that for every continuous

$$f = (f_1, \ldots, f_n) : X \to R^n,$$

for each $i \leq n$

$$f_i(A_i) \subset (-\infty, 0], \text{ and } f_i(B_i) \subset [0, \infty),$$

there exists $c \in X$ such that $f(c) = 0$.

$\{(A_i, B_i) : i = 1, \ldots, n\}$: *an n-dimensional boundary system.*

The n-dimensional Bolzano property

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The topological space X has *the n-dimensional Bolzano property* if there exists a family $\{(A_i, B_i) : i = 1, \dots, n\}$ of pairs of disjoint closed subsets such that for every family $\{(H_i^-, H_i^+) : i = 1, \dots, n\}$ of closed sets such that for each $i \leq n$

$$A_i \subset H_i^-, B_i \subset H_i^+ \text{ and } H_i^- \cup H_i^+ = X$$

we have

$$\bigcap\{H_i^- \cap H_i^+ : i = 1, \dots, n\} \neq \emptyset.$$

The n-dimensional Bolzano property

Theorem

If X has the n-dimensional Bolzano property. Then X has the Kulpa's n-dimensional Bolzano property.

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Theorem

If X is a normal and has the Kulpa's n-dimensional Bolzano property. Than X has the n-dimensional Bolzano property.

Properties

Theorem

Let $\{(A_i, B_i) : i = 1, ..., n\}$ *be the n-dimensional boundary system in* T_5 *space* X*. Then for each* $i_0 \in \{1, \ldots, n\}$
A_{i_0}, B_{i_0} *have an* $(n-1)$*-dimensional Bolzano property.*
Moreover the families

$$\{(A_{i_0} \cap A_i, A_{i_0} \cap B_i) : i \neq i_0\}, \{(B_{i_0} \cap A_i, B_{i_0} \cap B_i) : i \neq i_0\}$$

are an $(n-1)$*-dimensional boundary systems in* A_{i_0}, B_{i_0} *respectively.*

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Corollary

Let $I_1, I_2 \subset \{1, \ldots, n\}$, $I_1 \cap I_2 = \emptyset$*. Then the subspace*

$$\bigcap_{i \in I_1} A_i \cap \bigcap_{i \in I_2} B_i$$

has an $(n - (card(I_1) + card(I_2)))$*-dimensional Bolzano property.*

An n-cube-like polyhedron

Let A be a finite set.

Definition

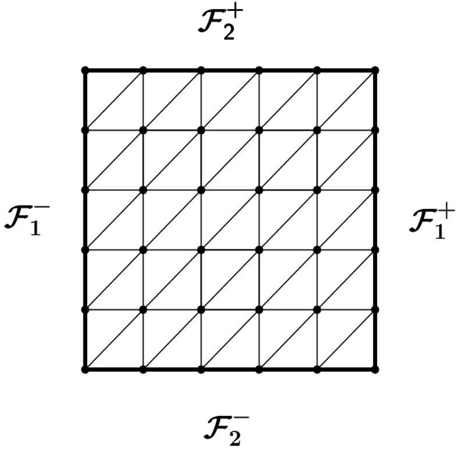
All complexes consisting of a single vertex are 0-cube-like ($\mathcal{K}^0$). The complex $\mathcal{K}^n$ generated by the family $\mathcal{S} \subset P_{n+1}(A)$ is said to be an *n-cube-like complex* if:

(A) for every $(n-1)$-face $T \in \mathcal{K}^n \setminus \partial\mathcal{K}^n$ there exists exactly two n-simplexes $S, S' \in \mathcal{K}^n$ such that $S \cap S' = T$.

(B) there exists a sequence of n pairs of subcomplexes $\mathcal{F}_i^-, \mathcal{F}_i^+$ called *i-th opposite faces* such that:

- (B_1) $\partial\mathcal{K}^n = \bigcup_{i=1}^n \mathcal{F}_i^- \cup \mathcal{F}_i^+$,
- (B_2) $\mathcal{F}_i^- \cap \mathcal{F}_i^+ = \emptyset$ for $i \in \{1, \ldots, n\}$,
- (B_3) for each $i_0 \in \{1, \ldots, n\}$ and each $\epsilon \in \{-, +\}$, $\mathcal{F}_{i_0}^\epsilon$ is an $(n-1)$-cube-like complex such that its opposite faces have a form $\mathcal{F}_{i_0}^\epsilon \cap \mathcal{F}_i^-, \mathcal{F}_{i_0}^\epsilon \cap \mathcal{F}_i^+$ for $i \neq i_0$.

An n-cube-like polyhedron



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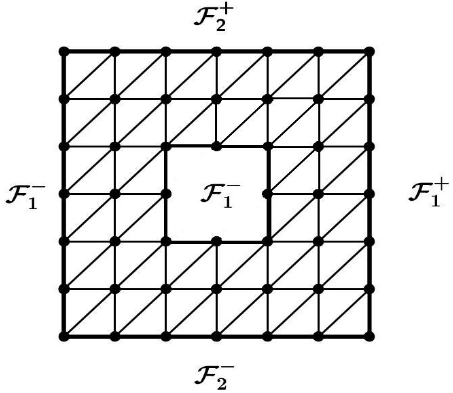

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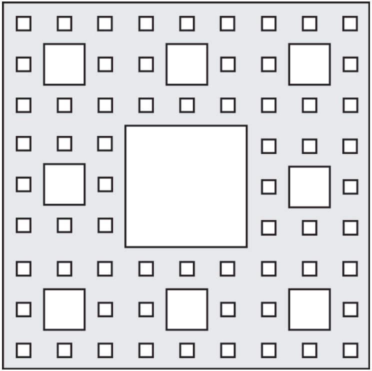

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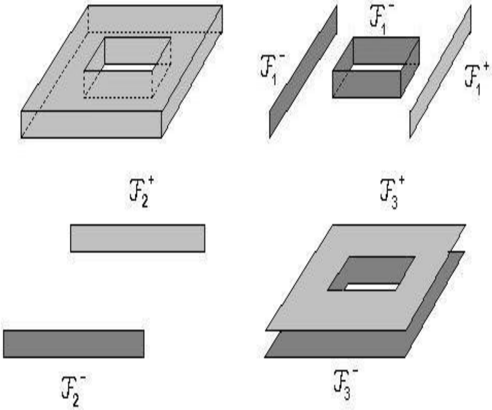

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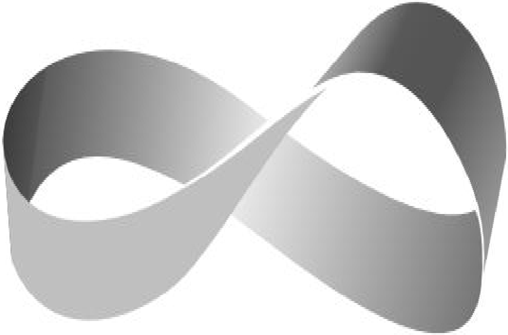

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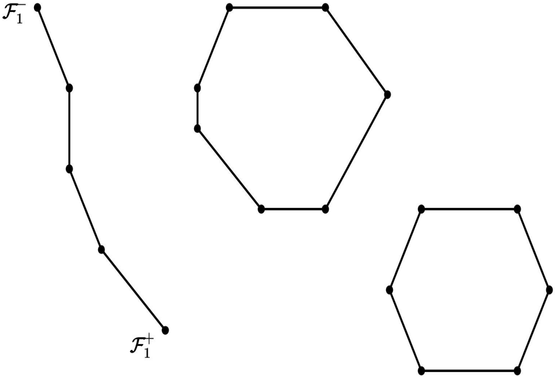

An n-cube-like polyhedron

Theorem

Let $(\bar{K}, \bar{\mathcal{K}})$ be an n-cube-like polyhedron in R^m. Then $\bar{K}$ has an n-dimensional Bolzano property.

The Steinhaus chains

Theorem (PT and Turzański 2008)

For an arbitrary decomposition of n-dimensional cube I^n onto k^n cubes and an arbitrary coloring function $F\colon T(k) \to \{1, ...n\}$ for some natural number $i \in \{1, ...n\}$ there exists an i-th colored chain $P_1, ..., P_r$ such that

$$P_1 \cap I_i^+ \neq \emptyset \text{ and } P_r \cap I_i^- \neq \emptyset.$$

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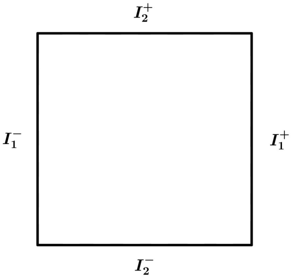

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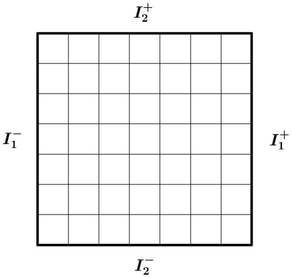

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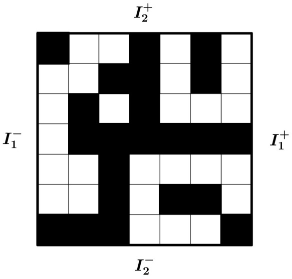

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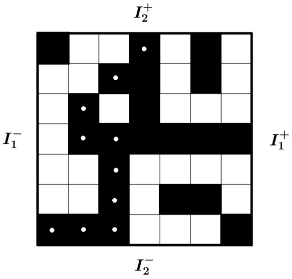

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Theorem (Topological version)

Let $\{U_i : i = 1, \cdots, n\}$ be an open covering of I^n. Then for some $i \in \{1, ...n\}$ there exists continuum $W \subset U_i$ such that

$$W \cap I_i^- \neq \emptyset \neq W \cap I_i^+.$$

The Steinhaus chains

Theorem (PT and Turzański)

The following statements are equivalent:

1. *Theorem(on the existence of a chain)*
2. *The Poincaré theorem*
3. *The Brouwer Fixed Point theorem.*

The Steinhaus chains

Theorem (Michalik, P T, Turzański 2015)

Let $\mathcal{K}^n$ *be an n-cube-like complex. Then for every map* $\phi\colon |\mathcal{K}^n| \to \{1, ..., n\}$ *there exist* $i \in \{1, ..., n\}$ *and i-th colored chain* $\{s_1, ..., s_m\} \subset |\mathcal{K}^n|$ *such that*

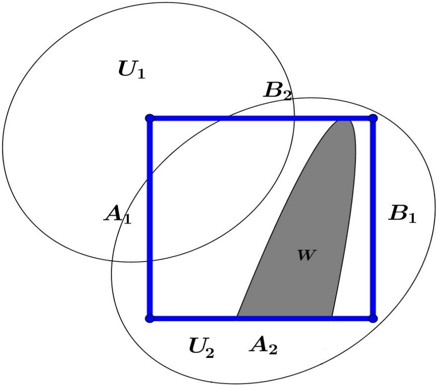
$$s_1 \in \mathcal{F}_i^- \text{ and } s_m \in \mathcal{F}_i^+.$$

(The sequence $\{s_1, ..., s_m\} \subset |\mathcal{K}^n|$ is a *chain* if for each $i \in \{1, ..., m-1\}$ we have $\{s_i, s_{i+1}\} \in \mathcal{K}^n$.)

Characterization of the Bolzano property

Theorem

Let X be a locally connected space. A family $\{(A_i, B_i) : i = 1, \ldots, n\}$ of pairs of disjoint closed subsets is an n-dimensional boundary system iff for each open covering $\{U_i\}_{i=1}^{n}$ for some $i \leq n$ there exists a connected set $W \subset U_i$ such that $W \cap A_i \neq \emptyset \neq W \cap B_i$.



The inverse system

Let us consider the inverse system $\{X_\sigma, \pi^\sigma_\rho, \Sigma\}$ where:
(i) $\forall \sigma \in \Sigma$ X_σ is a compact space with n-dimensional boundary system $\{(A^\sigma_i, B^\sigma_i) : i = 1, ..., n\}$.
(ii) $\forall \sigma, \rho \in \Sigma, \rho \leq \sigma$ the map $\pi^\sigma_\rho : X_\sigma \to X_\rho$ is a surjection such that $\pi^\sigma_\rho(A^\sigma_i) = A^\rho_i, \ \pi^\sigma_\rho(B^\sigma_i) = B^\rho_i$.

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Theorem

The space $X = \varprojlim\{X_\sigma, \pi^\sigma_\rho, \Sigma\}$ *has n-dimensional Bolzano property.*

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Theorem

The space $X = \varprojlim\{X_\sigma, \pi^\sigma_\rho, \Sigma\}$ *has n-dimensional Bolzano property.*

Corollary

Pseudo-arc has the Bolzano property.

The Bolzano property and the dimension

Theorem (on Partitions)

Let X be a normal space. $dim X \geq n$ iff there exists a family $\{(A_i, B_i) : i = 1, \ldots, n\}$ of pairs of non-empty disjoint closed subsets such that for every family $\{L_i : i = 1, \ldots, n\}$ where L_i is a partition between A_i and B_i we have

$$\bigcap_{i=1}^{n} L_i \neq \emptyset.$$

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Theorem

If a normal space X has n-dimensional Bolzano property. Then $dimX \geq n$.

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Theorem

If a normal space X has n-dimensional Bolzano property. Then $\dim X \geq n$.

Theorem

If $X \times [0,1]$ is a normal space X and $\dim X \geq n$. Then X has an n-dimensional Bolzano property.

Problem

Is there a gap between the Bolzano property and the dimension of X?