Dehn filling of a Hyperbolic 3-manifold

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Plan of the talk

1. Background of hyperbolic 3-manifolds.
2. Dehn filling.
3. Dehn parental test.

Background

Definition: A hyperbolic 3-manifold is a quotient $\mathbb{H}^3/\Gamma$ of three-dimensional hyperbolic space $\mathbb{H}^3$ by a subgroup Γ of hyperbolic isometries $PSL(2,\mathbb{C})$ acting freely and properly discontinuously.

The subgroup Γ is isomorphic to the fundamental group $\pi_1(M)$.

Theorem (Mostow-Prasad Rigidity, '74)

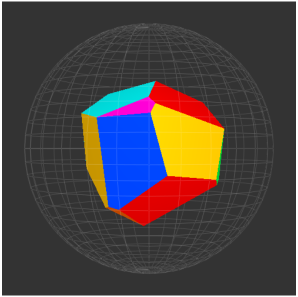
If M_1 and M_2 are complete finite volume hyperbolic n-manifolds, $n > 2$, any isomorphism of fundamental groups $\varphi : \pi_1(M_1) \to \pi_1\ (M_2)$ is realized by a unique isometry.

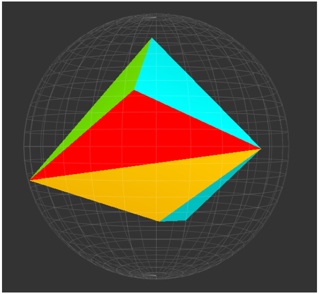
Geometric invariants (volume, geodesic length) are topological invariants.

Thurston, Jorgensen (1977) gave classification of finite volume hyperbolic 3-manifolds by their volume.

Background

M is a complete finite volume hyperbolic 3-manifold:
- closed
- cusped





Dirichlet domains of closed and cusped hyperbolic 3-manifolds from SnapPy

Background

Every element $\gamma \in \Gamma$ corresponds to a closed geodesic $g \subset M$.

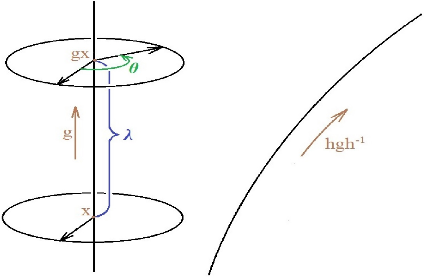
Every preimage of g in $\mathbb{H}^3$ is preserved by γ or its conjugates.

Definition: *Complex length* $l(\gamma)$ of a closed geodesic g in a hyperbolic 3-manifold is a number $\lambda + i\theta$,
λ is a geodesic's length and a minimal distance of transformation γ,
θ is the angle of rotation incurred by traveling once around γ, defined modulo 2π.

Definition: *Length spectrum* $L(M)$ of a hyperbolic 3-manifold is the set of complex length of all closed geodesics in M taken with multiplicities:

$$L(M) = \{l(\gamma) | \forall \gamma \in \Gamma\} \subset \mathbb{C}.$$

It is a discrete ordered set.



Dehn Filling

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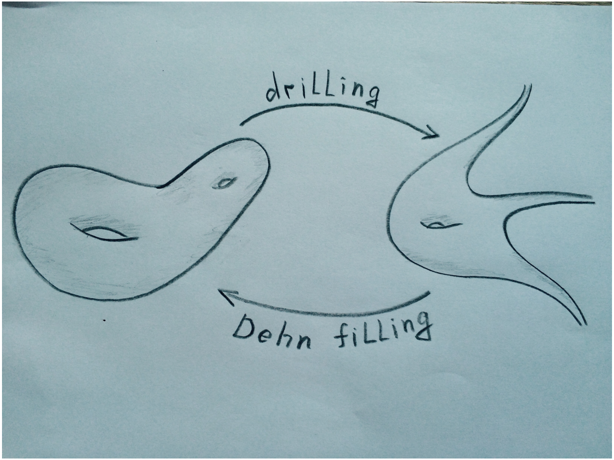
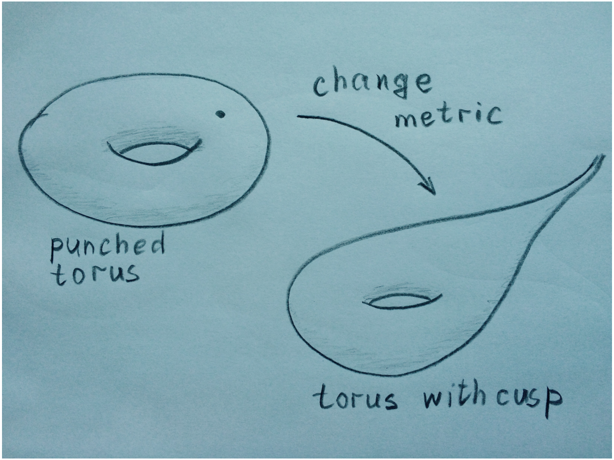


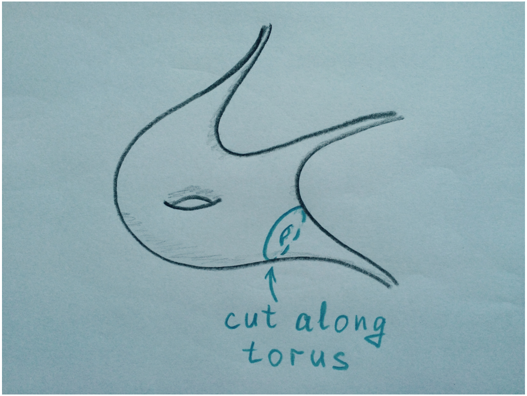
Dehn Filling

Drilling in dimension 2:



Dehn Filling

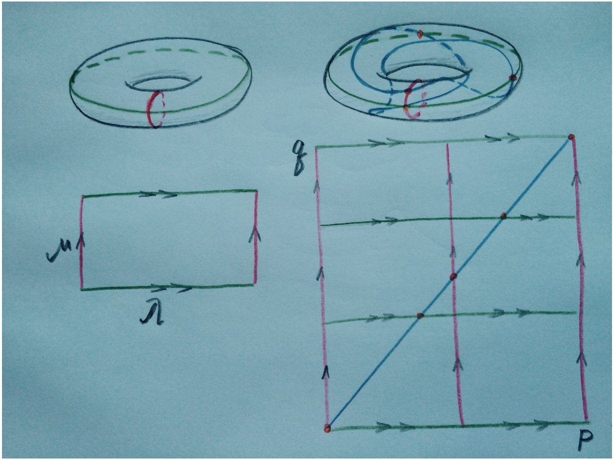
M - complete finite volume hyperbolic 3-manifold,
$\partial M = \sqcup T_i$
Dehn filling of M - "compactification".



Glue back solid torus with a Dehn twist. Result not always a manifold.

Dehn Filling

Framing of each T_i: set of meridians and longitudes (μ, λ).



Definition: *Slope* is an isotopy class of unoriented essential simple closed curves in the boundary of M.

Slope is identified with element of $\mathbb{Q} \cup \infty$ via $p/q \leftrightarrow \pm(p\mu + q\lambda)$.

Dehn Filling

Theorem (Thurston's Dehn Surgery Theorem, 1970's)

Let M - compact, orientable 3-manifold,
$\partial M = \sqcup T_i$ - finite number of tori components,
interior of M - admits complete, finite volume hyperbolic metrics.
Then ALL BUT A FINITE number of filling curves on each T_i give a closed 3-manifold with hyperbolic structure (otherwise we have "exceptional curves").

Question: How many exceptional fillings a manifold M has?

Answer: At most 10 for 1-cusped manifolds (M.Lackenby - R.Meyerhoff, 2008).

Dehn Parental Test

M, N - orientable 3-manifolds, admit complete hyperbolic metrics of finite volume on their interiors.

Question: Is N a Dehn filling of M?

Dehn Parental Test

C.Hodgson - S.Kerckhoff (2008) described the first practical method for determining Dehn filling heritage.

Theorem (R.Haraway, 2015)

Let M, N be orientable 3-manifolds admitting complete hyperbolic metrics of finite volume on their interiors. Let $\Delta V = Vol(M) - Vol(N)$. N is a Dehn filling of M if and only if either:

- *N is a Dehn filling of M along a slope c of normalized length $L(c) \leq 7.5832$, or*
- *N has a closed simple geodesic γ of length $l(\gamma) < 2.879\Delta V$ and N is a Dehn filling of M along a slope c such that*

$$4.563/\Delta V \leq L^2(c) \leq 20.633/\Delta V.$$

Dehn Parental Test

Dehn parental test for hyperbolic 3-manifolds reduces to rigorous calculations of

- volume (HIKMOT in Python, 2013),
- length spectra (Ortholength.nb, D.Gabai-M.T., 2012),
- cusp area,
- slope length (fef.py by B.Martelli-C.Petronio-F.Roukema, 2011, K.Ichihara-H.Masai, 2013),
- isometry test (SnapPea by J.Weeks).

Work in progress:

- write a rigorous algorithm for length spectra in Python using interval arithmetic.
- combine all existing programs to perform Dehn parental test as one command in SnapPy.

Conclusion

Dehn parental test:

- allows to determine Dehn filling heritage between two hyperbolic 3-manifolds
- can be verified rigorously with computer programs.

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THANK YOU!