There are no large sets which can be translated away from every Marczewski null set

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Theorem (Brendle-W., 2015)

(ZFC) No set of reals of size continuum is "s_0-shiftable".

Definition

A set $Y \subseteq 2^\omega$ is Marczewski null ($Y \in s_0$) $:\Longleftrightarrow$
for any perfect set $P \subseteq 2^\omega$ there is a perfect set $Q \subseteq P$ with $Q \cap Y = \emptyset$.

$$\Longleftrightarrow \forall p \in \mathbb{S} \quad \exists q \leq p \quad [q] \cap Y = \emptyset$$

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A set $X \subseteq 2^\omega$ is s_0-shiftable $:\Longleftrightarrow \forall Y \in s_0 \quad X + Y \neq 2^\omega$
$\Longleftrightarrow \forall Y \in s_0 \quad \exists t \in 2^\omega \quad (X + t) \cap Y = \emptyset.$

Theorem (Brendle-W., 2015, restated more explicitly)

(ZFC) Let $X \subseteq 2^\omega$ with $|X| = \mathfrak{c}$. Then there is a $Y \in s_0$ with $X + Y = 2^\omega$.

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Strong measure zero

For an interval $I \subseteq \mathbb{R}$, let $\lambda(I)$ denote its length.

Definition (well-known)

A set $X \subseteq \mathbb{R}$ is (Lebesgue) measure zero if
for each positive real number $\varepsilon > 0$
there is a sequence of intervals $(I_n)_{n<\omega}$ of total length $\sum_{n<\omega} \lambda(I_n) \leq \varepsilon$
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$\mathcal{M}$	σ-ideal of meager sets
$\mathcal{N}$	σ-ideal of Lebesgue measure zero ("null") sets
s_0	σ-ideal of Marczewski null sets

$\mathcal{M}$-shiftable	$\iff$	strong measure zero
$\mathcal{N}$-shiftable	$:\iff$	strongly meager
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only the countable sets are $\mathcal{M}$-shiftable	$:\iff$	BC
only the countable sets are $\mathcal{N}$-shiftable	$:\iff$	dBC
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Consistency of MBC

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Corollary

CH implies MBC (i.e., s_0-shiftables $= [2^\omega]^{\leq\aleph_0}$).

So what about larger continuum?

Theorem (Brendle-W., 2015)

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Properties of s_0

Proposition

Let $Y \subseteq 2^\omega$ with $|Y| < \mathfrak{c}$. Then $Y \in s_0$.

Why? Perfect sets can be split into "perfectly many" disjoint perfect sets.

Theorem

There is a set $Y \in s_0$ with $|Y| = \mathfrak{c}$.

Sketch of proof.

- Fix a maximal antichain $\{q_\alpha : \alpha < \mathfrak{c}\} \subseteq \mathbb{S}$ in Sacks forcing.
- In particular, $|[q_\alpha] \cap [q_\beta]| \leq \aleph_0$ for any $\alpha \neq \beta$.
- So (for each $\alpha < \mathfrak{c}$) we can pick $y_\alpha \in [q_\alpha] \setminus \bigcup_{\beta<\alpha}[q_\beta]$.
- By maximality of the antichain, and the proposition above, $Y := \{y_\alpha : \alpha < \mathfrak{c}\}$ is as desired. □

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Lemma

Let $X \subseteq 2^\omega$, and let $D \subseteq \mathbb{S}$ be a dense and translation-invariant set of Sacks trees with the property that any less than c many (of its bodies) do not cover X.
Then there is a $Y \in s_0$ such that $X + Y = 2^\omega$ (i.e., X is not s_0-shiftable).

Sketch of proof.

- Fix a maximal antichain $\{q_\alpha : \alpha < \mathfrak{c}\} \subseteq D$ (within the dense set D).
- Fix an enumeration $2^\omega = \{z_\alpha : \alpha < \mathfrak{c}\}$.
- By our assumptions, we can pick $x_\alpha \in X \setminus \bigcup_{\beta<\alpha}(z_\alpha + [q_\beta])$.
- Let $y_\alpha := x_\alpha + z_\alpha$. And let $Y := \{y_\alpha : \alpha < \mathfrak{c}\}$.
- Then
  - $X + Y = 2^\omega$, and
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Transitive versions of s_0

$$Y \in s_0 \iff \forall p \in \mathbb{S} \quad \exists q \leq p \qquad [q] \cap Y = \emptyset$$

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Let $X \subseteq 2^\omega$ with $|X| = \mathfrak{c}$.
Then there exists an $X' \subseteq X$ with $|X'| = \mathfrak{c}$ and $X' \in s_0$.

Recall the notion of Luzin set (we could say: $\mathcal{M}$-Luzin set):

X is (generalized) Luzin if
$(|X| = \mathfrak{c}$ and) its intersection with any meager set is of size less than $\mathfrak{c}$.

So the above lemma says:

There are no "s_0-Luzin sets" (in ZFC).

Proof.

- 1st case: $X \in s_0$, and we are finished :-)
- 2nd case: $X \notin s_0$, then we can fix $p \in \mathbb{S}$ with: $\forall q \leq p \quad |[q] \cap X| = \mathfrak{c}$.
  Construct a $\mathfrak{c}$-sized set $X' \in s_0$ inside of $[p] \cap X$:
  Fix a maximal antichain below p, ..., ..., and we are finished :-) □

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Let $X \subseteq 2^\omega$, and let $D \subseteq \mathbb{S}$ be a dense and translation-invariant set of Sacks trees with the property that any less than $\mathfrak{c}$ many (of its bodies) do not cover X.
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Main Lemma (more complicated, but not stronger!)

Assume $\mathfrak{c}$ is singular. Let $X \subseteq 2^\omega$ with $|X| = \mathfrak{c}$.
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Skew perfect sets in an arbitrary (?) Polish group G

Definition

$Z \subseteq G$ is skew if for all $x, y, z, w \in Z$ with $x \neq y$, $z \neq w$, and $\{x, y\} \neq \{z, w\}$, we have $x - y \neq z - w$.

Proposition

Assume $Z \subseteq G$ is skew and $t \in G$ with $t \neq 0$. Then $|Z \cap (Z + t)| \leq 2$.

Proposition

Being skew is translation-invariant.

Lemma

The skew perfect sets are dense in the perfect sets,
i.e., for each perfect set $P \subseteq G$ there is a skew perfect set $Q \subseteq P$.

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