HYPERSPACES OF COMPACT CONVEX SETS AND THEIR ORBIT SPACES

Sergey A. Antonyan

National University of Mexico

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Some Motivation

For every $n \geq 1$, let us denote:

- $cc(\mathbb{R}^n)$ the hyperspace of all compact convex subsets of $\mathbb{R}^n$,
- $cb(\mathbb{R}^n)$ the hyperspace of all compact convex bodies of $\mathbb{R}^n$,

equipped with the Hausdorff metric topology:

$$d_H(A, B) = \max \left\{ \sup_{b \in B} d(b, A), \sup_{a \in A} d(a, B) \right\},$$

where d is the Euclidean metric and $d(b, A) = \inf\{d(b, a) \mid a \in A\}$.

Theorem (Nadler, Quinn, and Stavrakas (1979))

1. *For $n \geq 2$, $cc(\mathbb{R}^n)$ is homeomorphic to $Q \setminus \{pt\}$, where $Q = [0,1]^{\aleph_0}$, the Hilbert cube,*
2. *For $n \geq 2$, $cc(\mathbb{B}^n)$ is homeomorphic to the Hilbert cube Q, where $\mathbb{B}^n$ stands for the closed unit ball of $\mathbb{R}^n$.*

Question

1. *What is the topological structure of $cb(\mathbb{R}^n)$, $n \geq 2$?*
2. *What is the topological structure of $cb(\mathbb{B}^n)$, $n \geq 2$?*

Theorem (S. Antonyan and N. Jonard-PÃ©rez (2013))

$cb(\mathbb{R}^n)$ is homeomorphic to $Q \times \mathbb{R}^{n(n+3)/2}$.

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$cb(\mathbb{R}^n)$ is homeomorphic to $Q \times \mathbb{R}^{n(n+3)/2}$.

Affine group action on $cb(\mathbb{R}^n)$

Question: Why is important to study $cb(\mathbb{R}^n)$ and its orbit spaces?

Answer: $cb(\mathbb{R}^n)/Aff(n) \cong BM(n)$ – the Banach-Mazur compactum.

Lets recall $BM(n)$.
In his 1932 book *Théorie des Opérations Linéaires*, S. Banach introduced the space of isometry classes $[X]$, of n-dimensional Banach spaces X equipped with the well-known Banach-Mazur metric:

$$d([X],[Y]) = \log \inf \left\{ \|T\| \cdot \|T^{-1}\| \mid T : X \to Y \text{ a linear isomorphism} \right\}$$

$$BM(n) = \{[X] \mid \dim X = n\}$$

the Banach-Mazur compactum.

- It is a challenging open problem whether $BM(n) \cong Q$, $n \geq 3$?
- It is known that $BM(2) \ncong Q$ (Ant., Fund Math. 2002)

Our approach is largely based on the study of the natural affine group action $Aff(n) \curvearrowright cb(\mathbb{R}^n)$.

$\mathrm{Aff}(n)$ is the group of all non-singular affine transformations of $\mathbb{R}^n$.

$g \in \mathrm{Aff}(n)$ iff $g(x) = v + \sigma(x)$ for every $x \in \mathbb{R}^n$, where $\sigma \in GL(n)$ and v is a fixed vector.

Definition

For a topological group G and a space X, an action $G \curvearrowright X$ is a continuous map

$$G \times X \to X, \quad (g, x) \mapsto gx$$

such that

- $(g \cdot h)x = g(hx)$
- ex=x

for all $g, h \in G$, e – the identity of G, and $x \in X$.

- It is a challenging open problem whether $BM(n) \cong Q$, $n \geq 3$?
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For $x \in X$, the orbit is $G(x) = \{gx \mid g \in G\}$.

$$X/G = \{G(x) \mid x \in X\}$$

denotes the orbit set.

$p : X \to X/G$, $p : x \mapsto G(x)$, is the orbit map.
X/G, equipped with the quotient topology, is called orbit space.

$\mathrm{Aff}(n)$ acts on $cb(\mathbb{R}^n)$ by the following rule:

$$\mathrm{Aff}(n) \times cb(\mathbb{R}^n) \to cb(\mathbb{R}^n)$$

$$(g, A) \mapsto gA = \{g(a) \mid a \in A\}.$$

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The action $Aff(n) \curvearrowright cb(\mathbb{R}^n)$ is proper.

Definition (Palais, 1961)

An action of a locally compact Hausdorff group G on a Tychonoff space X is proper if every point $x \in X$ has a neighborhood V_x such that for any point $y \in X$ there is a neighborhood V_y with the property that the *transporter from V_x to V_y*

$$\langle V_x, V_y \rangle = \{g \in G \mid gV_x \cap V_y \neq \emptyset\}$$

has compact closure in G.

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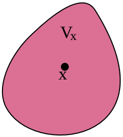

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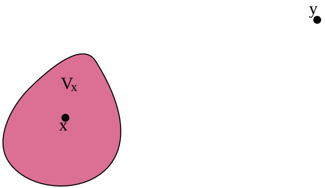

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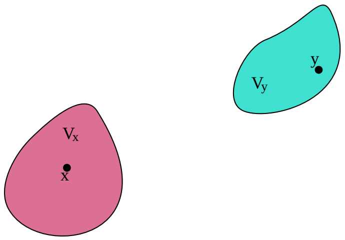

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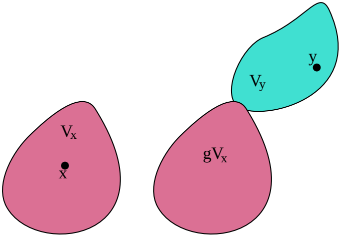

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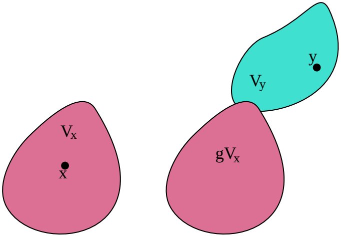

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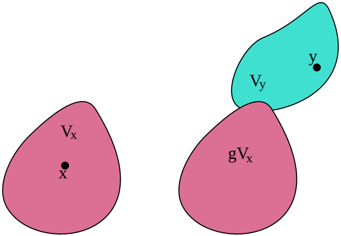

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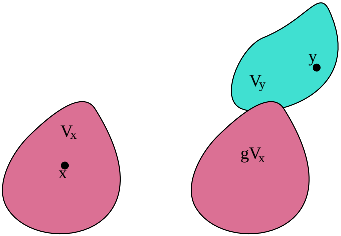

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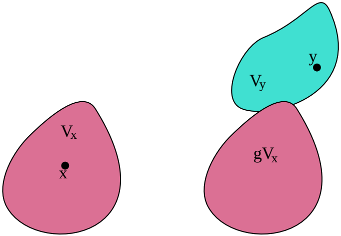

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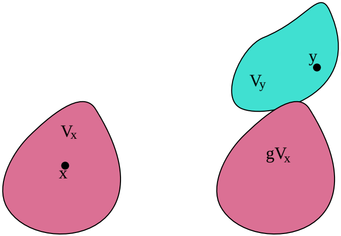

Theorem

1. *The action* $Aff(n) \curvearrowright cb(\mathbb{R}^n)$ *is proper.*
2. *There exists a global* $O(n)$*-slice* S *for* $cb(\mathbb{R}^n)$.
3. $cb(\mathbb{R}^n) \cong S \times Aff(n)/O(n)$.

Where comes the number $n(n+3)/2$ from?
in the above mentioned result:

$$cb(\mathbb{R}^n) \cong Q \times \mathbb{R}^{n(n+3)/2}.$$

Answer:

$$Aff(n)/O(n) \cong \mathbb{R}^{n(n+3)/2}.$$

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To obtain the final result

$$cb(\mathbb{R}^n) \cong Q \times \mathbb{R}^{n(n+3)/2},$$

it remains to find
a convenient $O(n)$-slice S for $cb(\mathbb{R}^n)$ such that $S \cong Q$.

Global Slices

Definition

Let $G := Aff(n)$, $H := O(n)$ and $X := cb(\mathbb{R}^n)$.
A subset $S \subset X$ is called a global *H-slice*, if the following conditions hold:

- $G(S) = X$, where $G(S) = \bigcup_{g \in G} gS$.
- S is closed in $G(S)$.
- S is H-invariant.
- $gS \cap S = \emptyset$ for all $g \in G \setminus H$.

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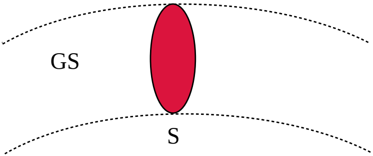

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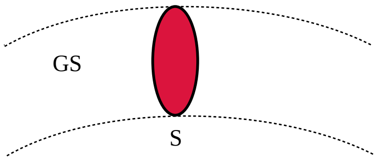

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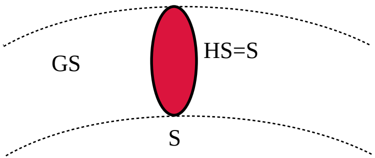

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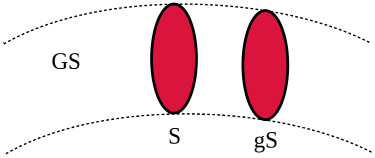
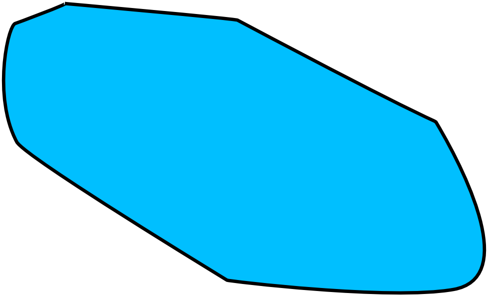


The John ellipsoid

For every compact convex body $A \in cb(\mathbb{R}^n)$ there exists a unique minimal volume ellipsoid $j(A)$ containing A. The ellipsoid $j(A)$ is called the John (sometimes also the Löwner) ellipsoid of A.



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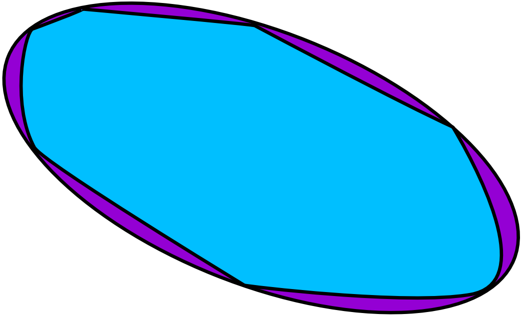
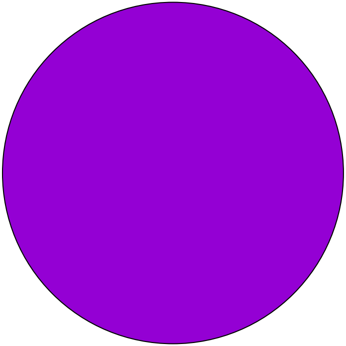


For every $n \geq 2$, lets denote by $J(n)$ the following set:

$$J(n) = \{A \in cb(\mathbb{R}^n) \mid j(A) = \mathbb{B}^n\}.$$



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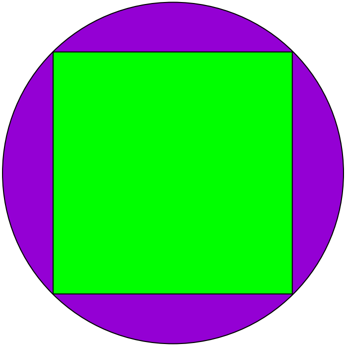

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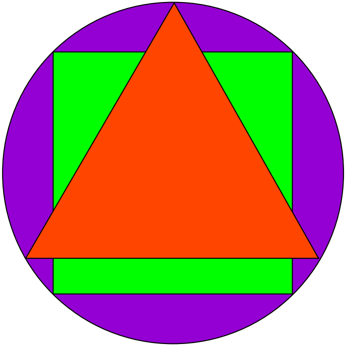

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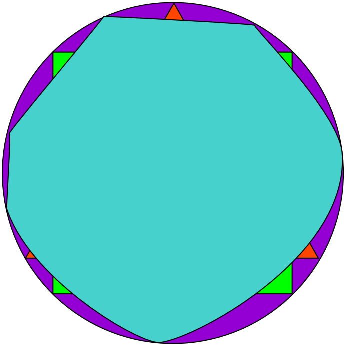

Theorem

$J(n)$ is a global $O(n)$-slice for the action $Aff(n) \curvearrowright cb(\mathbb{R}^n)$.

Hence,

$$cb(\mathbb{R}^n) \cong J(n) \times \mathbb{R}^{n(n+3)/2}.$$

Theorem

$J(n) \cong Q$.

Hiperspaces of $\mathbb{B}^n$

For every $n \geq 2$, we denote:

- $cc(\mathbb{B}^n)$ – the hyperspace of all compact convex subsets of $\mathbb{B}^n$,
- $cb(\mathbb{B}^n)$ – the hyperspace of all compact convex bodies of $\mathbb{B}^n$.

It is known that $cc(\mathbb{B}^n) \cong Q$ (Nadler et al).

But
What is $cb(\mathbb{B}^n)$?

Theorem

- $cb(\mathbb{B}^n) \cong Q \setminus \{*\}$.
- *Moreover, for any closed subgroup $K < O(n)$ that acts non-transitively on the unit sphere $\mathbb{S}^{n-1}$, the orbit space $cb(\mathbb{B}^n)/K \cong Q \setminus \{*\}$.*

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While the topological structure of the orbit space $cb(\mathbb{B}^n)/O(n)$ remains unknown,

for the orbit space $cc(\mathbb{B}^n)/O(n)$ we have the following

Theorem (Ant, Jonard-Pérez)

$$cc(\mathbb{B}^n)/O(n) \cong Cone(BM(n)).$$

Conjecture

$$cc(\mathbb{B}^n)/O(n) \not\cong Q.$$

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Another interesting geometrically defined hyperspaces are related to the Čebyshev ball. Recall that for any compact subset $A \subset \mathbb{R}^n$, there exists a unique ball $\check{C}(A)$ of minimum radius that contains A. It is called Čebyshev ball or circumball of A.

$$\check{c}(\mathbb{B}^n) := \{A \in cc(\mathbb{B}^n) \mid \check{C}(A) = \mathbb{B}^n\}.$$

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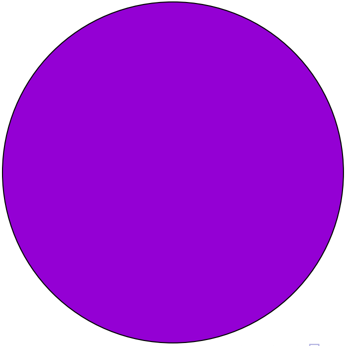

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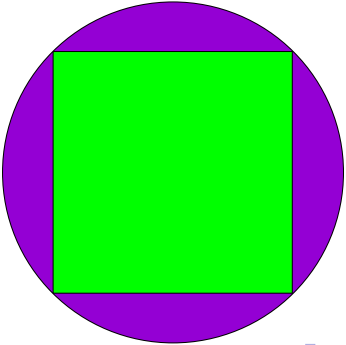

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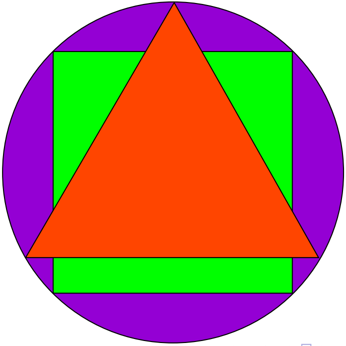

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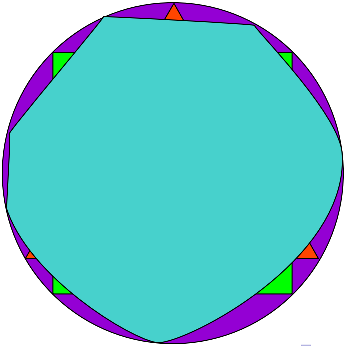

Where they come from?

Again consider the hyperspace $cb(\mathbb{R}^n)$. Now consider the natural action of the similarity group $Sim(n) \curvearrowright cb(\mathbb{R}^n)$.

Here $Sim(n) < Aff(n)$ and every $g \in Sim(n)$ is defined as

$$g(x) = u + t\sigma(x) \quad u \in \mathbb{R}^n, \quad \sigma \in O(n), \quad t > 0.$$

Since the action $Sim(n) \curvearrowright cb(\mathbb{R}^n)$ is proper, we have

Theorem

1. *$\check{c}b(\mathbb{B}^n)$ is a global $O(n)$-slice for the action $Sim(n) \curvearrowright cb(\mathbb{R}^n)$.*
2. $cb(\mathbb{R}^n) \cong \check{c}b(\mathbb{B}^n) \times Sim(n)/O(n)$.

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Since $Sim(n)/O(n) \cong \mathbb{R}^{n+1}$, we get

$$cb(\mathbb{R}^n) \cong \check{c}b(\mathbb{B}^n) \times \mathbb{R}^{n+1}.$$

From the other hand,

$$cb(\mathbb{R}^n) \cong J(n) \times \mathbb{R}^{n(n+3)/2},$$

Hence,

$$\check{c}b(\mathbb{B}^n) \times \mathbb{R}^{n+1} \cong J(n) \times \mathbb{R}^{n(n+3)/2} \cong Q \times \mathbb{R}^{n(n+3)/2}.$$

This makes me believe this

Conjecture

$$\check{c}b(\mathbb{B}^n) \cong Q \times \mathbb{R}^{(n+2)(n-1)/2}.$$

Theorem

1. $\check{c}(\mathbb{B}^n) \cong Q$,
2. $\check{c}b(\mathbb{B}^n)$ *is an open* $O(n)$*-invariant subset of the Hilbert cube* $\check{c}(\mathbb{B}^n)$,
3. *The complement* $\check{c}(\mathbb{B}^n) \setminus \check{c}b(\mathbb{B}^n)$ *is a* Z*-subset and*

$$\check{c}(\mathbb{B}^n) \setminus \check{c}b(\mathbb{B}^n) \cong \mathbb{RP}^{n-1}$$

Recall that a Z-set here means that for every $\varepsilon > 0$, there exists a continuous map

$$f : \check{c}(\mathbb{B}^n) \to \check{c}b(\mathbb{B}^n) \quad \text{such that} \quad d(f(A), A) < \varepsilon, \quad \forall \ A \in \check{c}(\mathbb{B}^n).$$

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As to the orbit spaces, we have the following

Theorem

For any closed subgroup $K < O(n)$ that acts non-transitively on the unit sphere $\mathbb{S}^{n-1}$,

1. $\check{c}(\mathbb{B}^n)/K \cong Q$,
2. $\check{c}b(\mathbb{B}^n)/K$ *is an open* $O(n)$*-invariant subset of the Hilbert cube* $\check{c}(\mathbb{B}^n)/K$ *whose complement* $\check{c}(\mathbb{B}^n) \setminus \check{c}b(\mathbb{B}^n)$ *is a* Z*-subset.*
3. $\check{c}(\mathbb{B}^n)/O(n) \cong BM(n)$,

The End

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