Compact spaces associated to Banach lattices

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- $C(K)$, $L^p(\mu)$ with $f \leq g$ iff $f(x) \leq g(x)$ for (almost) all x.
- Spaces with unconditional basis with coordinatewise order: ℓ_2, ℓ_p....

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Theorem (Lotz 1969, Schaefer, Kakutani)

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$=$ How do separable Banach lattices look like as vector lattices?

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  $E = C(2^{\mathbb{N}}, L_1[0,1]) \longrightarrow$ surjective universal sick compactum
  $E = Free(\mathbb{N}) \longrightarrow$ injective universal sick compactum

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Every measure in a Rosenthal compactum is of countable type and analytic. Converse true if K is separable .

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A lot of $\beta\mathbb{N}$

Theorem

If K is sick, then $\exists M_1 \subseteq M_2 \subseteq \cdots K$ closed metrizable, such that if $x_i \notin M_i$ discrete, then $\overline{\{x_n\}} = \beta\mathbb{N}$.

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Proof: $T : C(K) \hookrightarrow E$,

$$M_n = \{t \in K : \|Tf\| \geq 2^{-n} \text{ for } |f| \leq 1, f(t) = 1\} \quad \square$$

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Corollary

$\beta\mathbb{N} \times \beta\mathbb{N}$ is not sick.

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- A is (hereditarily in B)$^\perp$ iff $0 \notin \overline{\{\|x_n\| : n \in A\}}$.
- (hereditarily in B)$^\perp$ is a countably generated ideal.
- $\beta\mathbb{N} \times 2^{\mathbb{N}}$ is not sick.

Non-pathological analytic P-ideals

These are ideals of the form

$$\mathscr{I} = \left\{ A \subseteq \mathbb{N} : \lim_m \sup_{c \in C} \sum_{n \in A, n \geq m} c_n = 0 \right\} \text{ for } C \subset c_{00} \cap \ell_1^+$$

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- Density 0 ideal.

$$\mathscr{I} = \left\{ A \subset \mathbb{N} : \lim_n \frac{|A \cap \{1, \ldots, n\}|}{n} = 0 \right\}$$

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Theorem (Borodulin-Nadzieja, Farkas + Plebanek)

$\mathscr{J}$ is a non-pathological P-ideal iff
there is an unconditional basis $\{e_n\}$ such that

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We add $e = \sup e_n$ to their space, similarly as one does with c_0 to obtain c.

$$(t_1, t_2, \dots) = \sup\{\sum c_i |t_i| : c \in C\}, \quad E = \langle e_n, (1,1,1,\dots)\rangle$$