Automatic continuity of measurable homomorphisms on Čech-complete topological groups

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Motivation: the Cauchy equation

A function $f : \mathbb{R} \to \mathbb{R}$ is

- *additive* if $\forall x, y \in \mathbb{R} \quad f(x+y) = f(x) + f(y)$;
- *linear* if $\quad \exists a \in \mathbb{R} \quad \forall x \in \mathbb{R} \quad f(x) = ax$.

Theorem (Cauchy, 1821)

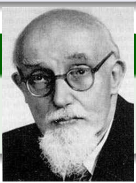
A function $f : \mathbb{R} \to \mathbb{R}$ is linear iff it is additive and continuous.

Problem (Cauchy)

Are there any nonlinear (and hence discontinuous) solutions of the Cachy function equation $f(x+y) = f(x) + f(y)$?

Example (Hamel, 1905)

The Axiom of Choice implies the existence of $2^{\mathfrak{c}}$ discontinuous additive functions $f : \mathbb{R} \to \mathbb{R}$.

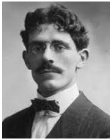


Lebesgue measurable additive functions are continuous

Theorem (Fréchet, 1913; Banach, Sierpiński, 1920)

Any Lebesgue-measurable additive function $f:\mathbb{R}\to\mathbb{R}$ *is continuous*

M. Fréchet, *Pri la funckcia ekvacio* $f(x+y)=f(x)+f(y)$, L'Ensenement Mathematique, **15** (1913), 390–393.



W. Sierpiński, *Sur léquation fonctionnelle* $f(x+y)=f(x)+f(y)$, Fundamenta Mathematica **1** (1920), 116–122.

S. Banach, *Sur l'équation fonctionnelle* $f(x+y)=f(x)+f(y)$, Fundamenta Mathematicae **1** (1920), 123–124.

The reason: Steinhaus-Weil Theorem

Theorem (Steinhaus, 1920)

For any subset $A \subseteq \mathbb{R}$ of positive Lebesgue measure, the set $A - A = \{x - y : x, y \in A\}$ is a neighborhood of zero.

H. Steinhaus, *Sur les distances des points dans les ensembles de mesure positive*, Fundamenta Mathematicae **1** (1920), 93–104.

Theorem (Weil, $\leq$ 1940)

For any Haar-measurable subset A of positive Haar measure in a locally compact topological group X the set $AA^{-1} = \{xy^{-1} : x, y \in A\}$ is a neighborhood of the identity.

A. Weil, *L'intégration dans les groupes topologiques*, Hermann, 1940.

Definition: A *Haar measure* on a topological group X is any nonzero left-invariant σ-additive measure $\lambda : \mathcal{B}o(X) \to [0, \infty]$ on the σ-algebra $\mathcal{B}o(X)$ of Borel subsets of X such that

- each compact subset K of X has finite measure $\lambda(K)$ and
- for every Borel set $B \subseteq X$ and every real number $a < \lambda(B)$ there exists a compact set $K \subseteq B$ such that $\lambda(K) > a$.

Theorem (Haar, 1933)

Every locally compact group has a Haar measure and such a measure is unique up to a multiplicative constant.

A. Haar, *Der Massbegriff in der Theorie der kontinuierlichen Gruppen*, Annals of Mathematics, **34**:1 (1933), 147–169.

Theorem (Weil, 1936)

A topological group has a Haar measure iff it is locally compact.

ω-narrow topological groups

Definition (Arhangel'skiĭ–Guran–Tkachenko, 1981)

A topological group X is *ω-narrow* if for any nonempty open set $U \subseteq X$ exists a countable set $C \subseteq X$ such that $UC = X = CU$.

Theorem (Guran, 1981)

- *A topological group X is ω-narrow iff X is a subgroup of the Tychonoff product of second-countable topological groups.*
- *The class of ω-narrow topological groups is closed under taking Tychonoff products, subgroups, continuous homomorphic images and contains all topological groups which are separable or Lindelöf or countably cellular (or has any other reasonable countability property).*

I. Guran, *Topological groups similar to Lindelöf groups*, Dokl. Akad. Nauk SSSR. **256**:6 (1981), 1305–1307.

Definition: A function $f : X \to Y$ from a locally compact topological group X to a topological space Y is called *Haar-measurable* if for every open set $U \subseteq Y$ the preimage $f^{-1}[U]$ is measurable with respect to the Haar measure λ on X.
The latter means that there exist two Borel sets B, B' in X such that $B \subseteq f^{-1}[U] \subseteq B'$ and $\lambda(B' \setminus B) = 0$.

Theorem (folklore, probably known to Weil in 40-ies)

Every Haar-measurable homomorphism $h : X \to Y$ from a locally compact topological group X to any ω-narrow topological group Y is continuous.

Continuity of Haar-homomorphisms into ω-narrow groups

Theorem (folklore, probably known to Weil in 40-ies)

Every Haar-measurable homomorphism $h : X \to Y$ from a locally compact group X to an ω-narrow group Y is continuous.

Proof: Given any nbhd U of the identity e_Y in Y, choose a nbhd $V \subseteq Y$ of e_Y such that $VV^{-1} \subseteq U$. By the Haar-measurability of h, the set $A = h^{-1}[V]$ is Haar-measurable in X. By ω-narrowness of $h[X] \subseteq Y$, there is a countable $C \subseteq X$ such that $Y = h[C] \cdot V$. Then $X = C \cdot h^{-1}[V] = CA$ and $\lambda(A) > 0$ by the σ-additivity of the Haar measure λ on X. By Steinhaus–Weil, AA^{-1} is a neighborhood of e_X. Then $h[AA^{-1}] = h[A]h[A]^{-1} \subseteq VV^{-1} \subseteq U$ and hence $h^{-1}[U] \supseteq AA^{-1}$ is a nbhd of e_X, so h is continuous.

Problem (folklore, probably known to Weil in 40-ies)

Can the ω-narrowness be removed from this theorem?

Theorems of Kleppner and Kuznetsova

Theorem (Kleppner, 1989–91)

Any Haar-measurable homomorphism between locally compact groups is continuous.

- A. Kleppner, *Measurable homomorphisms of locally compact groups*, PAMS **106** (1989), 391–395.
- A. Kleppner, *Correction to: "Measurable homomorphisms of locally compact groups"*, PAMS **111** (1991) 1199–1200.

Theorem (Kuznetsova, 2012)

Under Martin's Axiom, every Haar-measurable homomorphism from a locally compact group to any top group is continuous.

- Y. Kuznetsova, *On continuity of measurable group representatio[ns] and homomorphisms*, Studia Math. **210** (2012), 197–208.

The first main result: The continuity of Haar-measurable homomorphisms

Theorem (B., 2021)

Every Haar-measurable homomorphism from a locally compact topological group to an arbitary topological group is continuous.

Well, this seems to be a final result. So, what next?

What does happen for measurable homomorphisms on arbitarary (not necessarily locally compact) topological groups?

In this case no Haar measures exist, so we cannot talk about Haar-measurability.

A natural substitute for the Haar-measurability in the non-locally compact case is the universally measurability.

Definition: A Radon measure on a topological space X is any nonzero σ-additive Borel measure $\lambda : \mathcal{B}o(X) \to [0, \infty]$ such that

- each compact subset K of X has finite measure $\lambda(K)$ and
- for every Borel set $B \subseteq X$ and every real number $a < \lambda(B)$ there exists a compact set $K \subseteq B$ such that $\lambda(K) > a$.

Definition: A subset A of a topological space X is called *universally measurable* if A is measurable with respect to any Radon measure λ on X.

The latter means that there exist two Borel subsets B, B' of X such that $B \subseteq A \subseteq B'$ and $\lambda(B' \setminus B) = 0$.

Definition: A function $f : X \to Y$ between topological spaces is called *universally measurable* if the preimage $f^{-1}[U]$ of any open set $U \subseteq Y$ is universally measurable in X.

Continuity of universally measurable homomorphisms

Question (very naïve)

Is each universally measurable homomorphism between topological groups continuous?

No:

The homomorphism

$$h : \mathbb{Z} + \sqrt{2}\mathbb{Z} \to \mathbb{Z} + \sqrt{3}\mathbb{Z}, \quad h : x + \sqrt{2}y \mapsto x + \sqrt{3}y$$

between the countable dense subgroups $\mathbb{Z} + \sqrt{2}\mathbb{Z}$ and $\mathbb{Z} + \sqrt{3}\mathbb{Z}$ of the real line is universally measurable but discontinuous.

Well, what about homomorphisms between Polish groups?

Problem (Christensen, 1971)

Is each universally measurable homomorphism between **Polish** groups continuous?

Continuity of universally measurable homomorphisms

A topological group X is called a *balanced* if each neighborhood of the identity in X contains a neighborhood U, which is *balanced* is the sense that $xU = Ux$ for all $x \in X$.

It is clear that each commutative topological group is balanced.

Theorem (Christensen, 1971)

Any universally measurable homomorphism from a Polish topological group to a balanced ω-narrow topological group is continuous.

J.P.R. Christensen, *Borel structures in groups and semigroups*, Math. Scand. **28** (1971), 124–128.

18.4.1944--1.7.2022

Definition (Christensen): A universally measurable subset A of a topological group X is *left Haar-null* if there exists a Radon measure μ on X such that $\mu(xA) = 0$ for all $x \in X$.

Theorem (Christensen, 1971)

If a universally measurable subset A of a Polish group X is not left Haar-null, then there is a finite set $F \subseteq X$ such that $\bigcup_{x \in F} xAA^{-1}x^{-1}$ is a neighborhood of the identity in X. If the set A is balanced, then AA^{-1} is a neighborhood of the identity.

Corollary (Christensen, 1971)

Any universally measurable homomorphism from a Polish topological group to a balanced ω-narrow topological group is continuous.

The problem of Christensen was eventually resolved by Christian Rosendal who removed the balanced condition from Christensen's theorem. This step took almost 50 years!

Theorem (Rosendal, 2019)

Any universally measurable homomorphism from a Polish topological group to any ω-narrow topological group is continuous.

C. Rosendal, *Continuity of universally measurable homomorphisms*, Forum Math. Pi. **7** (2019), e5, 20 pp.

Question

Can the ω-narrowness be removed from Rosendal's theorem?

Yes!

And moreover, the Polishness can be replaced by the Čech-completeness!

The second main result: The continuity of universally measurable homomorphisms

Definition: A topological group is called Čech-complete if it is homeomorphic to a G_δ-subset of some compact Hausdorff space.

It is well-known that a metrizable space is Čech-complete if and only if its topology is generated by a complete metric.

So, each Polish group is Čech-complete.

Theorem (B., 2022)

Every universally measurable homomorphism $h : X \to Y$ from a Čech-complete topological group X to a topological group Y is continuous.

A subset A of a topological space X has the Baire Property in X if A belongs to the σ-algebra generated by open and meager sets.

It is well-known that a set $A \subseteq X$ has the Baire Property if and only if there exists an open set $U \subseteq X$ such that the symmetric difference $U \triangle A$ is meager ($=$ of the I-st Baire category) in X.

The following is a Baire category counterpart of Steinhaus-Weil.

Theorem (Piccard, 1939; Pettis, 1951)

For any **nonmeager** *set A with the Baire Property in a topological group X, the set AA^{-1} is a neighborhood of the identity in X.*

- S. Piccard, *Sur les ensembles de distances des ensembles de points d'un espace Euclidien*, Mém. Univ. Neuchâtel, **13** (1939), 212 pp.
- B.J. Pettis, *Remarks on a theorem of E.J. McShane*, Proc. Amer. Math. Soc. **2** (1951), 166–171.

Continuity of *BP*-measurable homomorphisms

Definition: A function $f : X \to Y$ between topological spaces is called *BP-measurable* if the preimage $f^{-1}[U]$ of any open set $U \subseteq Y$ the the Baire Property in X.

Piccard-Pettis Theorem implies the following

Theorem (math folklore, probably known to Pettis in 50-ies)

Every BP-measurable homomorphism from a nonmeager topological group to an ω-narrow topological group is continuous.

Problem

Can the ω-narrowness be removed from this theorem?

Partly Yes: the ω-narrowness can be moved from the range to the domain

The third main result: The continuity of BP-measurable homomorphisms

Theorem (B., 2021)

Every BP-measurable homomorpism $h : X \to Y$ from an ω-**narrow** Čech-complete topological group X to an arbitrary topological group Y is continuous.

Problem (persistent)

Can the ω-narrowness be removed from this theorem?

Partly Yes: for universally BP-measurable homomorphisms

The fourth main result: The continuity of universally *BP*-measurable homomorphisms

A subset A of a topological space X has the *universal Baire Property* if for every closed subspace $F \subseteq X$ the intersection $A \cap F$ has the Baire Property in F.

A function $f : X \to Y$ between topological spaces is called *universally BP-measurable* if the preimage $f^{-1}[U]$ of any open subset $U \subseteq Y$ has the universal Baire Property in X.

Theorem (B., 2021)

Every universally *BP*-measurable homomorpism from a Čech-complete topological group to an arbitrary topological group is continuous.

Summary: Continuity of homomorphisms on Čech-complete groups

Corollary (B., 2022)

For a homomorphism $h : X \to Y$ from a Čech-complete topological group X to an arbitrary topological group Y, the following conditions are equivalent:

1. *h is continuous;*
2. *h is Borel;*
3. *h is universally measurable;*
4. *h is universally BP-measurable.*

Problem

Can the *BP*-measurability of h be added to the above list?
Partial answer: Yes if X or Y is ω-narrow.

What is behind the curtain?

Theorem (B., 2021)

1. *Every Haar-measurable homomorphism from a locally compact topological group to any topological group is continuous.*
2. *Every BP-measurable homomorphism from an ω-narrow Čech-complete group to any topological group is continuous.*

These two results are corollaries of a single

Theorem (B., 2021)

Let X be a nonmeager K-analytic topological group and $\mathcal{I}$ is a ccc left-invariant σ-ideal with functionally Borel base on X such that $\mathcal{B}o(X) \subseteq \mathcal{B}a(X)^{\pm\mathcal{I}}$. A homomorphism $h : X \to Y$ to a topological group Y is continuous if and only if h is $Bo(X)^{\pm\mathcal{I}}$-measurable.

Let us explain the concepts and notions appearing in this theorem.

K-analytic topological groups

Definition. A topological space (or group) X is *K-analytic* if X is a continuous image of a Lindelöf Čech-complete space.

A metrizable space is K-analytic if and only if it is analytic, i.e., a continuous image of a Polish space.

Theorem (B., 2021)

For a topological group X the following conditions are equivalent:

1. *X is nonmeager and K-analytic;*
2. *X is ω-narrow and Čech-complete;*
3. *X is Lindelöf and Čech-complete;*
4. *X is countably cellular and Čech-complete;*
5. *X has a balanced compact subgroup K such that the quotient group X/K is Polish.*

Functionally Borel sets in topological spaces

Definition: A subset A of a topological space X is called *functionally arbitrary* (resp. *functionally Borel*) if $A = f^{-1}[B]$ for some continuous map $f : X \to \mathbb{R}^\omega$ and some (Borel) set $B \subseteq \mathbb{R}^\omega$.

Let $\mathcal{B}o(X)$ (resp. $\mathcal{B}a(X)$) be the σ-algebra of all (functionally) Borel sets in a topological space X.

σ-ideals

A family $\mathcal{I}$ of subsets of a set X is called a *σ-ideal* on X if for any countable subfamily $\mathcal{C} \subseteq \mathcal{I}$, any subset $C \subseteq \bigcup \mathcal{C}$ belongs to $\mathcal{I}$.

Let $\mathcal{I}$ be a σ-ideal on a topological space X. We say that $\mathcal{I}$

- has a (*functionally*) *Borel base* if each set $I \in \mathcal{I}$ is a subset of some (functionally) Borel set $B \in \mathcal{I}$ in X;
- is ccc if every disjoint subfamily $\mathcal{B}o(X) \setminus \mathcal{I}$ is countable.

Let $\mathcal{B}a(X)^{\pm\mathcal{I}} \overset{def}{=} \{A \subseteq X : \exists B \in \mathcal{B}a(X) \ \ (A \Delta B \in \mathcal{I})\}$ be the σ-algebra generated by $\mathcal{B}a(X) \cup \mathcal{I}$.

Example

1. The σ-ideal $\mathcal{M}$ of meagers subsets of a countably cellular Tychonoff space is ccc and has functionally Borel base.
2. The σ-ideal $\mathcal{N}$ of sets of Haar-measure zero in an ω-narrow locally compact topological group is ccc and has a functionally Borel base.

A σ-ideal $\mathcal{I}$ on a group X is *left-invariant* if $\{xI : I \in \mathcal{I},\ x \in X\} \subseteq \mathcal{I}$.
A set $A \subseteq X$ is *$\mathcal{I}$-positive* if $A \notin \mathcal{I}$.

Theorem (ideal version of Steinhaus–Weil–Piccard–Pettis)

Let $\mathcal{I}$ be a left-invariant ccc σ-ideal with a functionally Borel base on an ω-narrow Čech-complete top group. For any $\mathcal{I}$-positive set $A \in \mathcal{B}a(X)^{\pm}\mathcal{I}$ the set $AA^{-1}AA^{-1}$ is a nbd of the identity in X.

Proof.

Since $\mathcal{I}$ has a functionally Borel base, the $\mathcal{I}$-positive set $A \in \mathcal{B}a(X)^{\pm}\mathcal{I}$ contains an $\mathcal{I}$-positive functionally Borel (and hence K-analytic) subset B. Let $M \subseteq X$ be a maximal set such that the family $(xB)_{x\in M}$ is disjoint. The ccc property of $\mathcal{I}$ ensures that M is countable and the maximality of M implies that $X = MBB^{-1}$. Since $X = MBB^{-1}$ is nonmeager, the K-analytic set BB^{-1} is not meager in X and by the Piccard–Pettis Theorem, $BB^{-1}BB^{-1} \subseteq AA^{-1}AA^{-1}$ is a neighborhood of the unit. □

Continuity of $\mathcal{B}a(X)^{\pm}\mathcal{I}$-measurable homomorphisms

Corollary

Let $\mathcal{I}$ be a left-invariant ccc σ-ideal with a functionally Borel base on an ω-narrow Čech-complete top group X. Every $\mathcal{B}a(X)^{\pm}\mathcal{I}$-measurable homomorphism $h : X \to Y$ to an ω-narrow topological group Y is continuous.

Now the trick is to remove the requirement of ω-narrowness of Y from this theorem.

This will be done with the help of the notion of the $\mathcal{A}^{\curlyvee}\mathcal{I}$-semimeasurability and a suitable theorem about nonmeasurable unions.

$\mathcal{A}^\Upsilon\mathcal{I}$-semimeasurability

Definition

Let $\mathcal{I}$ be a σ-ideal on a topological space X. A subset $S \subseteq X$ is called *$\mathcal{A}^\Upsilon\mathcal{I}$-semimeasurable* if for any K-analytic set $A \subseteq X$ with $A \cap S \notin \mathcal{I}$, there exists an $\mathcal{I}$-positive K-analytic set $B \subseteq A \cap S$.

The following proposition shows that the measurability decomposes into two semimeasurabilities.

Proposition

Let $\mathcal{I}$ be a ccc σ-ideal with a functionally Borel base on a K-analytic space X such that $\mathcal{B}o(X) \subseteq \mathcal{B}a(X)^{\pm}\mathcal{I}$. A set $S \subseteq X$ belongs to the σ-ideal $\mathcal{B}o(X)^{\pm}\mathcal{I}$ if and only if S and $X \setminus S$ are $\mathcal{A}^\Upsilon\mathcal{I}$-semimeasurable.

Theorem (Brzuchowski, Cichoń, Grzegorek, Ryll-Nardzewski, 1979)

Let $\mathcal{I}$ be a σ-ideal with a Borel base on a Polish space X. Any point-finite family $\mathcal{J} \subseteq \mathcal{I}$ with $\bigcup \mathcal{J} \notin \mathcal{I}$ contains a subfamily $\mathcal{J}' \subseteq \mathcal{J}$ such that $\bigcup \mathcal{J}' \notin \mathcal{B}o(X)^{\pm}\mathcal{I}$.

Corollary

Let $\mathcal{I}$ be a left-invariant σ-ideal with a Borel base on a Polish group X. For any $\mathcal{B}o(X)^{\pm}\mathcal{I}$-measurable homomorphism $h : X \to Y$ to an arbitrary topological group Y and any neighborhood $U \subseteq Y$ of the identity we have $h^{-1}[U] \notin \mathcal{I}$. If $\mathcal{I}$ is ccc, then h is continuous.

"Analytic" generalization of the Four Poles Theorem

Theorem (B., Rałowski, Żeberski, 2021)

Let $\mathcal{I}$ be a σ-ideal on a Polish space X. Any point-finite family $\mathcal{J} \subseteq \mathcal{I}$ with $\bigcup \mathcal{J} \notin \mathcal{I}$ contains a subfamily $\mathcal{J}' \subseteq \mathcal{I}$ such that $\bigcup \mathcal{J}'$ is not $\mathcal{A}^{\curlyvee}\mathcal{I}$-semimeasurable.

A function $f : X \to Y$ to a topological space X is called *$\mathcal{A}^{\curlyvee}\mathcal{I}$-semimeasurable* if the preimage $f^{-1}[U]$ of any open set $U \subseteq Y$ is $\mathcal{A}^{\curlyvee}\mathcal{I}$-semimeasurable in X.

Corollary (B., 2021)

Let $\mathcal{I}$ be a left-invariant σ-ideal with a functionally arbitrary base on an ω-narrow Čech-complete topological group X.
For any $\mathcal{A}^{\curlyvee}\mathcal{I}$-semimeasurable homomorphism $h : X \to Y$ to an arbitrary topological group Y and any neighborhood $U \subseteq Y$ of the identity we have $h^{-1}[U] \notin \mathcal{I}$. If $\mathcal{I}$ is ccc, then h is continuous.

Two important corollaries

Corollary (B., 2021)

Let $\mathcal{I}$ be a left-invariant σ-ideal with a functionally arbitrary base on an ω-narrow Čech-complete topological group X.
For any $\mathcal{A}^{\curlyvee}\mathcal{I}$-semimeasurable homomorphism $h : X \to Y$ to an arbitrary topological group Y and any neighborhood $U \subseteq Y$ of the identity we have $h^{-1}[U] \notin \mathcal{I}$. If $\mathcal{I}$ is ccc, then h is continuous.

Applying this corollary to the σ-ideals $\mathcal{N}$ and $\mathcal{M}$, we obtain

Corollary (B., 2021)

1. *Any Haar-measurable homomorphism on a locally compact group is continuous.*
2. *Any BP-measurable homomorphism on an ω-narrow Čech-complete group is continuous.*

The bibliography

T. Banakh, *Automatic continuity of measurable homomorphisms on Čech-complete topological groups*, preprint (`arxiv.org/abs/2206.02481`).

Thank you for your attention!

and

for comments, questions, etc...

28 July 988 is the day of baptization of Kyiv Rus
and 28 July also is the official day of Ukrainian Statehood


СЛАВА УКРАЇНІ!
ГЕРОЯМ СЛАВА!