Completeness and topologizability of countable semigroups

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Theorem (Kuratowski-Mrówka)

A topological space X is compact if and only if for every space Y the natural projection $p : X \times Y \to Y$ is a closed map.

Definition (Dikranjan, Uspenskij)

A topological group G is called c-compact if for any topological group H the natural projection $G \times H \to H$ sends closed subgroups to closed subgroups.

It can be checked that every continuous homomorphic image of a c-compact topological group is Raikov complete.

Problem (Dikranjan, Uspenskij, 1998)

Is any c-compact topological group compact?

There are some positive partial solutions of this problem (by Banakh, Dikranjan, Lukách, Uspenskij and others). However, in general case this problem was solved in negative by Klyachko, Olshanskii and Osin.

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Solution

Expectation

The counterexample is constructed using some sophisticated topological techniques. Also, it possesses some strong compact-like property (sequential compactness, countable compactness, etc.)

Reality (Theorem by Klyachko, Olshanskii and Osin, 2013)

There exists a discrete bounded countable c-compact group G.

This example possesses an extremely exotic property. Namely any homomorphic image of any subgroup of G is nontopologizable, i.e., admits only the discrete group topology.
Thus, the solution is purely algebraic!

Aim of this talk

To develop a connection between nontopologizability and completeness. In particular, to show that in some sense the mentioned example is natural.

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Definitions

Definition

Let $\mathcal{C}$ be a class of topological semigroups containing all discrete semigroups. A semigroup X is called

- $\mathcal{C}$-nontopologizable if the only topology τ such that $(X,\tau) \in \mathcal{C}$ is discrete;
- projetively $\mathcal{C}$-nontopologizable if each homomorphic image of X is $\mathcal{C}$-nontopologizable.

We shall consider the classes:

- TG of Tychonoff topological groups;
- $\mathsf{T_zS}$ of Tychonoff zero-dimensional topological semigroups;
- $\mathsf{T_2S}$ of Hausdorff topological semigroups;
- $\mathsf{T_1S}$ of T_1 topological semigroups.

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Definitions

- A group polynomial on a group G is a function $f\colon G \to G$ of the form $f(x) = a_0x^{\epsilon_1}a_1 \cdots x^{\epsilon_n}a_n$ for some elements $a_0, \ldots, a_n \in G$ and $\epsilon_i \in \{-1, 1\}$, $i \leq n$.
- A semigroup polynomial on a semigroup X is a function $f\colon X \to X$ of the form $f(x) = a_0xa_1 \cdots xa_n$ for some elements $a_0, \ldots, a_n \in X^1$.

Nontopologizability of groups and semigroups can be described in terms of corresponding Zariski topologies.

For a group G its

- group Zariski topology $\mathfrak{Z}^{\pm}_G$ is generated by the subbase consisting of the sets $\{f(x) \neq e_G\}$, where f is a group polynomial on G.

For a semigroup X its

- Zariski topology $\mathfrak{Z}_X$ is the topology on X generated by the subbase consisting of the sets $\{x \in X : f(x) \neq b\}$ and $\{x \in X : f(x) \neq g(x)\}$ where $b \in X$ and f, g are semigroup polynomials on X.
- Zariski T_1-topology $\mathfrak{Z}'_X$ is the topology on X generated by the subbase consisting of the sets $\{x \in X : f(x) \neq b\}$ where $b \in X$ and f is a semigroup polynomial on X.

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Characterizations of nontopologizability

The following theorems characterize countable nontopologizable (semi)groups in terms of Zariski topologies.

Theorem (Markov)

A countable group G is TG-nontopologizable if and only if the group Zariski topology $\mathfrak{Z}_G^{\pm}$ is discrete.

Theorem (Kotov-Taimanov)

A countable semigroup X is T_2S-nontopologizable if and only if the Zariski topology $\mathfrak{Z}_X$ is discrete.

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In many cases, completeness properties of various objects of General Topology and Topological Algebra can be characterized externally as closedness in ambient objects. For example, a metric space X is complete if and only if X is closed in any metric space containing X as a subspace. A uniform space X is complete if and only if X is closed in any uniform space containing X as a uniform subspace. A topological group G is Raikov complete if and only if it is closed in any topological group containing G as a subgroup.

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- $\mathcal{C}$-closed if for any isomorphic topological embedding $h\colon X \to Y$ to a topological semigroup $Y \in \mathcal{C}$ the image $h[X]$ is closed in Y;
- projectively $\mathcal{C}$-closed if any homomorphic image of X is $\mathcal{C}$-closed;
- injectively $\mathcal{C}$-closed if for any injective homomorphism $i\colon X \to Y$ to a topological semigroup $Y \in \mathcal{C}$ the image $i[X]$ is closed in Y;
- absolutely $\mathcal{C}$-closed if for any homomorphism $h\colon X \to Y$ to a topological semigroup $Y \in \mathcal{C}$ the image $h[X]$ is closed in Y.

Recall that the group constructed by Klyachko, Olshanskii and Osin is absolutely TG-closed. Moreover, we shall see that it is absolutely T_1S-closed.

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Diagram

Observe that the completeness-like properties from the previous slide depend only on the algebraic structure of a semigroup X.

For any semigroup X the following implications hold:

$$\begin{array}{ccc} X \text{ is absolutely } \mathcal{C}\text{-closed} & \Longrightarrow & X \text{ is injectively } \mathcal{C}\text{-closed} \\ \Downarrow & & \Downarrow \\ X \text{ is projectively } \mathcal{C}\text{-closed} & \Longrightarrow & X \text{ is } \mathcal{C}\text{-closed.} \end{array}$$

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$\mathcal{C}$-closedness of countable groups

A semigroup X is called polybounded if $X = \bigcup_{i=1}^{n} \{x \in X : f_i(x) = b_i\}$ for some elements $b_1, \ldots, b_n \in X$ and semigroup polynomials $f_1, \ldots, f_n$ on X.

Theorem (Banakh, B.)

For a countable group G the following conditions are equivalent:

- G is projectively $\mathsf{T}_1\mathsf{S}$-closed;
- G is $\mathsf{T}_1\mathsf{S}$-closed;
- G is $\mathsf{T}_z\mathsf{S}$-closed;
- G is polybounded.

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For a countable group G the following conditions are equivalent:

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Some applications

Theorem (Banakh, B.)

Each polybounded T_1 paratopological group is a topological group.

Theorem (Banakh, B.)

Each polybounded cancellative semigroup is a group.

Corollary

Each polybounded cancellative T_1 topological semigroup is a topological group.

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Theorem (Banakh, B.)

For a semigroup X the following conditions are equivalent:

- X is injectively T_1S-closed;
- X is T_1S-closed and T_1S-nontopologizable.

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The group constructed by Klyachko, Olshanskii and Osin, being bounded, is absolutely T_1S-closed.

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Nontopologizability and completeness

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Final characterization

Theorem (Banakh, B.)

For a countable group G the following conditions are equivalent:

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- G is absolutely $\mathrm{T_1S}$-closed;
- G is $\mathrm{T_1S}$-nontopologizable;
- G is projectively $\mathrm{T_1S}$-nontopologizable;
- the T_1 Zariski topology $\mathfrak{Z}'_G$ on G is discrete.

Thank You for attention!

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