Partitioning the real line into Borel sets

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TOPOSYM

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Theorem (Hausdorff, 1936)

There is a partition of $\mathbb{R}$ into $\aleph_1$ nonempty $F_{\sigma\delta}$ sets.

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The following are equivalent:

1. *There is a partition of $\mathbb{R}$ into $\aleph_1$ nonempty G_δ sets.*
2. *There is a partition of $\mathbb{R}$ into $\aleph_1$ nonempty $G_{\delta\sigma}$ sets.*
3. *$\mathbb{R}$ can be covered with $\aleph_1$ meager sets, i.e., $\mathrm{cov}(\mathcal{M}) = \aleph_1$.*

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There is a partition of $\mathbb{R}$ into $\aleph_1$ closed sets if and only if there is a partition into $\aleph_1$ F_σ sets.

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There is a partition of $\mathbb{R}$ into $\aleph_1$ closed sets if and only if there is a partition into $\aleph_1$ F_σ sets. Furthermore, the existence of such a partition is not implied by $\mathrm{cov}(\mathcal{M}) = \aleph_1$.

The starting point for me

To summarize what we've seen so far, all of the following implications hold in ZFC, and none of them reverses:

The Continuum Hypothesis

$\Rightarrow$ There is a partition of $\mathbb{R}$ into $\aleph_1$ closed / F_σ sets

$\Rightarrow$ There is a partition of $\mathbb{R}$ into $\aleph_1$ G_δ / $G_{\delta\sigma}$ sets

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Question:

What about partitions of $\mathbb{R}$ into more than $\aleph_1$ Borel sets?

What's different about bigger κ?

Recall from earlier the easy part of the Fremlin-Shelah theorem:

If $\mathrm{cov}(\mathcal{M}) = \aleph_1$, *there is a partition of* $\mathbb{R}$ *into* $\aleph_1$ G_δ *sets.*

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Theorem (Miller, 1989)

Consistently, $\mathfrak{c} > \aleph_2$ *and* $\mathbb{R}$ *cannot be partitioned into* $\aleph_2$ *Borel sets.*

The partition spectrum of a pointclass

Question:

For what uncountable cardinals κ is there a partition of $\mathbb{R}$ into precisely κ Borel sets? What does the set of all such κ look like? What about G_δ sets or closed sets, or other pointclasses of sets?

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for any uncountable Polish space X.

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We know that $\aleph_1, \mathfrak{c} \in \mathfrak{sp}(\mathrm{Borel})$, and it is consistent with $\neg$CH to have $\aleph_2 \notin \mathfrak{sp}(\mathrm{Borel})$. Can anything else be said?

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Assuming GCH *holds up to* $\max(C)$, *there is a ccc forcing extension in which* $C = \mathfrak{sp}(\mathrm{closed})$, *and furthermore, if* $\min(C) < \mu \notin C$, *then* $\mu \notin \mathfrak{sp}(\mathrm{Borel})$.

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The proof utilizes an "isomorphism-of-names" argument in order to exclude cardinals $\mu \notin C$ from $\mathfrak{sp}(\text{Borel})$.

Some corollaries

Corollary

Given any $A \subseteq \omega \setminus \{0\}$*, there is a forcing extension in which* $\mathfrak{sp}(\text{closed}) = \{\aleph_n : n \in A\} \cup \{\aleph_\omega, \aleph_{\omega+1}\}$.

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Let C be a countable set of uncountable cardinals such that

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Thus, given any $A \subseteq \omega \setminus \{0\}$, there is a forcing extension in which $\mathfrak{sp}(\mathrm{Borel}) = \{\aleph_n : n \in A\} \cup \{\aleph_1, \aleph_\omega, \aleph_{\omega+1}\}$.

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1. $\mathfrak{sp}(\text{Borel})$ is countable,
2. $\min(\mathfrak{sp}(\text{Borel})) = \aleph_1$,
3. $\mathfrak{sp}(\text{Borel})$ has a maximum with uncountable cofinality,
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Question

Which of these items represent essential features of $\mathfrak{sp}(\mathrm{Borel})$, and which just represent limitations of the techniques used to prove the theorems on the previous slides?

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The first item on our list simply represents a limitation of our proof technique:

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The second and third items on our list are necessary features of $\mathfrak{sp}(\mathrm{Borel})$, because $\aleph_1 \in \mathfrak{sp}(\mathrm{Borel})$ by Hausdorff's theorem, and $\mathfrak{c} = \max(\mathfrak{sp}(\mathrm{Borel}))$ has uncountable cofinality.

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Fix κ sets $\langle B_\alpha : \alpha < \kappa \rangle$ in this partition. Each B_α contains an uncountable Polish space K_α.

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$\mathfrak{sp}(\mathrm{Borel})$ *is closed under singular limits.*

Proof.

Suppose λ is a limit of cardinals $\langle \mu_\alpha : \alpha < \kappa \rangle$ in $\mathfrak{sp}(\mathrm{Borel})$, with $\kappa = \mathrm{cf}(\lambda) < \lambda$. Pick $\mu_\gamma \geq \kappa$, and fix a partition $\mathcal{P}$ of ω^ω into μ_γ Borel sets. Without loss of generality, assume all members of this partition are uncountable.

Fix κ sets $\langle B_\alpha : \alpha < \kappa \rangle$ in this partition. Each B_α contains an uncountable Polish space K_α. Partition K_α into μ_α Borel sets, and then replace each B_α in $\mathcal{P}$ with these μ_α sets and $B_\alpha \setminus K_\alpha$. □

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Suppose that $0^\dagger$ *does not exist. If* λ *is a singular cardinal with* $\mathrm{cf}(\lambda) = \omega$ *and* $\lambda \in \mathfrak{sp}(\mathrm{Borel})$*, then* $\lambda^+ \in \mathfrak{sp}(\mathrm{Borel})$*.*

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Is it consistent (relative to some large cardinal hypothesis) that there is a singular cardinal λ *with* $\lambda \in \mathfrak{sp}(\mathrm{Borel})$ *but* $\lambda^+ \notin \mathfrak{sp}(\mathrm{Borel})$*?*

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Is it consistent (relative to some large cardinal hypothesis) that there is a singular cardinal λ *with* $\lambda \in \mathfrak{sp}(\mathrm{Borel})$ *but* $\lambda^+ \notin \mathfrak{sp}(\mathrm{Borel})$*? In particular, is it consistent to have a partition of* $\mathbb{R}$ *into* $\aleph_\omega$ *Borel sets, but not* $\aleph_{\omega+1}$*?*

Partitioning a space into Polish spaces

For any space X, define

$$\mathfrak{par}(X) = \min\{|\mathcal{P}| : \mathcal{P} \text{ is a partition of } X \text{ into Polish spaces}\}.$$

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For the remainder of the talk, all ordinals are considered to carry the discrete topology.

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Lemma (B. & Miller, 2015)

If $0 < n < \omega$, *then* $\mathfrak{par}(\omega_n^\omega) = \aleph_n$.

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$$X_\beta = \beta^n \setminus \bigcup_{\alpha<\beta} \alpha^n$$

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Let $0 < n < \omega$. Then there is a continuous bijection $\omega_n^\omega \to \omega^\omega$ if and only if $\aleph_n \in \mathfrak{sp}(\mathrm{Borel})$.

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$\Rightarrow$: Let $\mathcal{P}$ be a partition of ω_n^ω into $\aleph_n$ Polish spaces, and suppose $f : \omega_n^\omega \to \omega^\omega$ is a continuous bijection. Then $\{f[X] : X \in \mathcal{P}\}$ is a partition of ω^ω into $\aleph_n$ Borel sets. $\square$

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Lemma (B., 2022)

If κ is an uncountable cardinal, then $\mathfrak{par}(\kappa^\omega) \geq \mathrm{cf}([\kappa]^\omega, \subseteq)$. In particular, $\mathfrak{par}(\omega_\omega^\omega) \geq \aleph_{\omega+1}$.

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- Beginning with GCH plus the generalized Chang Conjecture $(\aleph_{\omega+1}, \aleph_\omega) \to (\aleph_1, \aleph_0)$, which is consistent relative to a huge cardinal, and then adding $> \aleph_{\omega+1}$ Cohen reals results in a model in which $\mathrm{cf}([\omega_\omega]^\omega, \subseteq) = \aleph_{\omega+1} < \mathfrak{par}(\omega_\omega^\omega)$.

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Currently we know the answer only for $\alpha = 1$ and 2, and the answer is yes in both cases.

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The end

Thank you for listening