# A proof of the Tree Alternative Conjecture for the topological minor relation.

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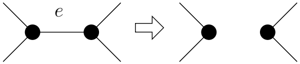
# Outline

- Background.
  - Operations on graphs.
  - Graph/tree relations.
  - WQOs.
- Tree Alternative Conjecture.
  - Generalised TAC.
  - A snippet of our proof.

# Local Operations on Graphs

1. Edge removal ($\setminus e$):



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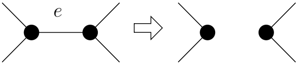


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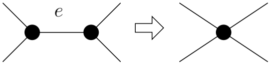

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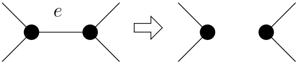


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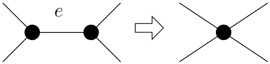


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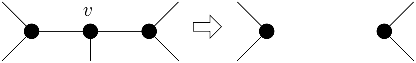

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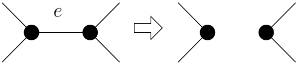


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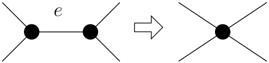


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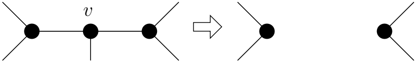
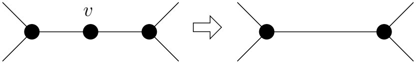



4. Deg-2 vertex dissolution ($/v$):



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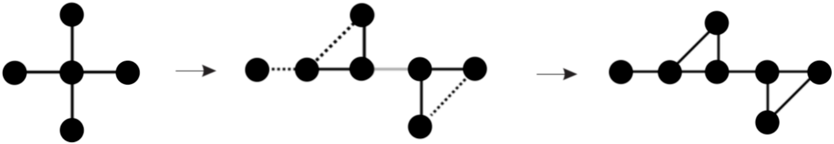
| Relation | $\backslash e$ | $/e$ | $\backslash v$ | $/v$ |
|---|---|---|---|---|
| subgraph/embeddable | • | | • | |
| induced subgraph/strongly embeddable | | | • | |
| topological minor | • | | • | • |
| induced topological minor | | | • | • |
| graph minor | • | • | • | |
| induced graph minor | | • | • | |

# Graph Relations

Graph minor



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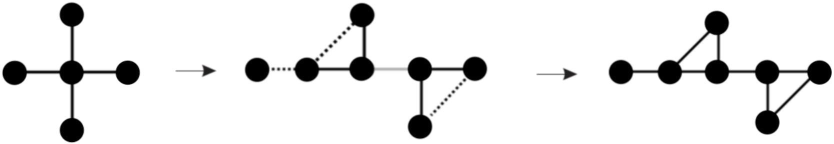
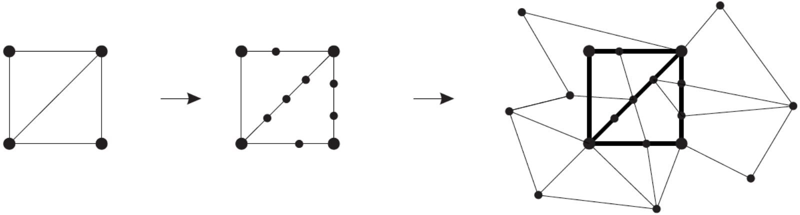


Topological Minor



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**Observation 3:** the topological and graph minor relations are *well-quasi-orders* on trees.

# WQOs

A **quasi-order** (QO) on a set $X$ is a reflexive and transitive relation $\leqslant$ and it becomes a **well-quasi-order** (WQO) if all strictly descending chains and antichains are finite.

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**Lemma**

*For a qo $(X, \leqslant)$ TFAE:*

1. $(X, \leqslant)$ *is a WQO.*
2. *For any sequence $(x_n)$ in $X$ there exists $i < j$ with $x_i \leqslant x_j$.*
3. *Any sequence $(x_n)$ in $X$ contains a monotone increasing subsequence.*

# WQOs on graphs

| Relation | WQO on (finite) trees | WQO on (finite) graphs |
|---|---|---|
| embeddable | (🔴) 🔴 | (🔴) 🔴 |
| topological minor | (🟢) 🟢 | (🔴) 🔴 |
| graph minor | (🟢) 🟢 | (🟢) 🔴 |

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- Nash-Williams proved the topological minor relation is a WQO on trees but the result can't be extended to all graphs. Consider the collection $A_n$ with:

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- The Graph Minor Theorem (Robertson and Seymour 84 - '87) shows that finite graphs are WQO under the graph minor using the powerful concept of *forbidden minors*.
- It is false for infinite graphs (Thomas '88) but true if at least one graph is planar and finite (Thomas '89).

# The Tree Alternative Conjecture

Trees $T$ and $S$ are *mutually embeddable*, $T \sim S$, is they can be embedded in each other.

The **Tree Alternative Conjecture (TAC)** states that - up to isomorphism - the number of trees mutually embeddable with a given tree is either 1 or infinite.

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- Solved for all rooted trees by Tyomkyn in 2008.
- Solved for *scattered trees* by Laflamme, Pouzet, and Sauer in 2017.

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**Theorem (B-Szeptycki, '22)**

*Up to isomorphism, the number of trees that are mutual topological/graph minors with a given tree is either 1 or infinite.*

The Theorem seems to be true for all graphs under either relation but this remains an open question.

**Conjecture**

*Up to isomorphism, the number of graphs that are mutual (induced) topological/graph minors with a given tree is either 1 or infinite.*

# Relative TAC

Finally, TAC can be asked of each relation relative to stronger ones: $\cong \geqslant \sim \geqslant \sim^{\sharp} \geqslant \sim^{*}$.

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For instance, letting $[T]_*$ denote the equivalence class of $T$ under $\sim^*$, what are the possible sizes for $[T]_*/\sim^\sharp$?

| | $\cong$ | $\sim$ | $\sim^\sharp$ | $\sim^*$ |
|---|---|---|---|---|
| $\cong$ | - | TAC (?) | ● | ● |
| $\sim$ | - | - | ● | ● |
| $\sim^\sharp$ | - | - | - | ? |
| $\sim^*$ | - | - | - | - |

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The dichotomy occurrs between *large* and *small* trees. A **small** trees is one where every ray is *eventually bare* and **large** if not small. A ray $R = v_1 v_2 \dots$ is **eventually bare** if $\exists k \in \mathbb{N}$ with $\deg(v_n) = 2$ for all $n \geqslant k$.

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**Theorem (B-Szeptycki, '21)**

*For any locally finite tree $T$: $|[T]_\sharp| = 1$ if $T$ is small and $2^{\aleph_0}$, otherwise.*

In fact (B-S '22) $T \cong S \iff T \sim^* S$ for locally finite small trees (i.e., all 4 relations coincide for small locally finite trees).

# TAC for tm

The neat dichotomy stopped with locally finite trees:

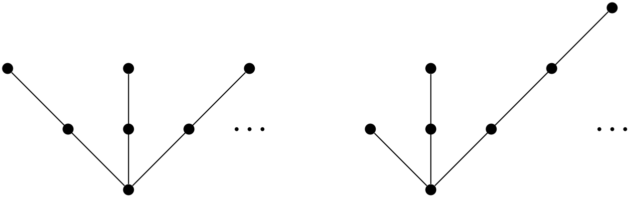

Figure: Left equivalence class $\aleph_0$ and right equivalence class $2^{\aleph_0}$.

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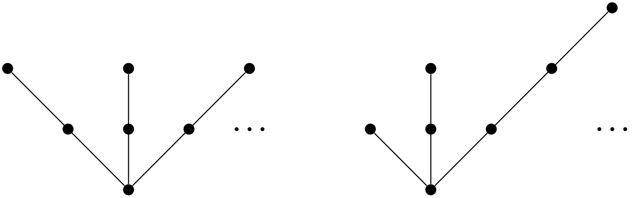

Figure: Left equivalence class $\aleph_0$ and right equivalence class $2^{\aleph_0}$.

**Theorem (B-Szeptycki, '22)**

*For any large tree $T$: $|[T]_\sharp| \geqslant 2^{\aleph_0}$.*

**Theorem (B-Szeptycki, '22)**

*For any small tree $T$: $|[T]_\sharp| = 1$ or $\geqslant \aleph_0$.*

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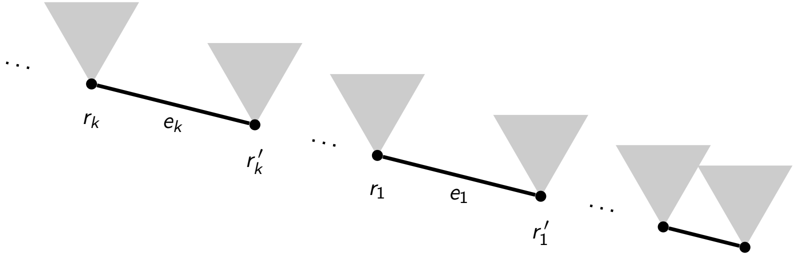
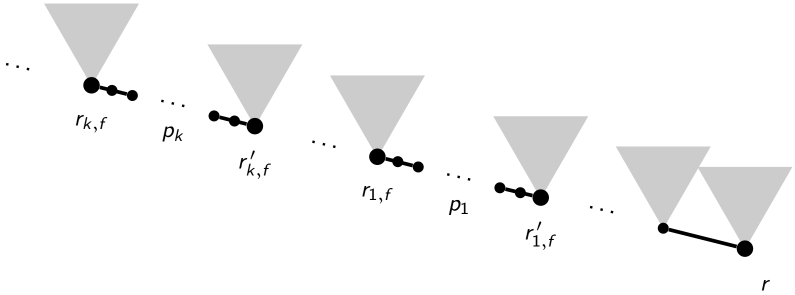
- Let $(T, r)$ be locally finite and large and find a ray $R = v_1 v_2 \dots$ that is not eventually bare.
- Consider the collection $T_n$ of full subtrees of $T$ rooted at $v_n$.
- B/c $\{T_n \mid n \in \mathbb{N}\}$ is WQO we can find an increasing subsequence $(r_k)$ of $(n)$ with $T_{r_i} \leqslant^\sharp T_{r_j}$ with $i \leqslant j$ and WLOG, $deg(r_k) \geqslant 3$.



# Large Trees and WQO

- For each $f : \mathbb{N} \to \mathbb{N}$ let $(T_f, r)$ denote the tree that results from the following subdivision of $(T, r)$: for each $n \in \mathbb{N}$ subdivide $e_n$ into a bare path of length $f(n)$.
- Let $p_n$ denote the bare path of length $f(n)$ that replaces $e_n$ in $(T, r)$ and $R_f$ the modified ray $R$.



# Large Trees and WQO

**Lemma**

*For any pair* $f, g \in \mathbb{N}^{\mathbb{N}}$, $(T_f, r) \equiv^{\sharp} (T_g, r)$.

**Proof.**

All $(T_f, r)$ are topologically equivalent to $(T, r)$.
Easy: $(T_f, r) \geqslant^{\sharp} (T, r)$.
Hard: $(T_f, r) \leqslant^{\sharp} (T, r)$ - but again all b/c $\sim^{\sharp}$ is a WQO of trees. □

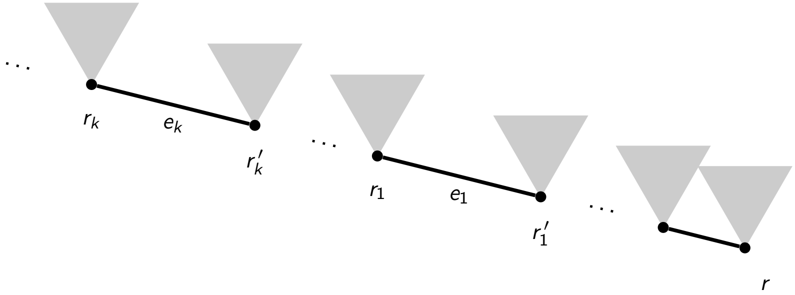
# Large Trees and WQO

For each $n \in \mathbb{N}$:

$$l_n = \{v \in v(T, r) \mid level(v) = level(r_n')\}$$

and

$$L_n = \{p : p \text{ is a finite maximal bare path with initial vertex } \in l_n\}.$$



# Large Trees and WQO

- For a path $p$ to be in $L_n$ it must be that if $v$ is the terminal vertex of $p$ then $deg(v) > 2$.
- Since $deg(r_k) > 2$ for all $k$ then $L_n \neq \emptyset$ for all $n$.
- Since $(T, r)$ is locally finite, it follows that $M_n = \max\{|p| : p \in L_n\}$ exists.

# Large Trees and WQO

**Lemma**

*Let* $f, g \in \mathbb{N}^{\mathbb{N}}$ *so that* $f(n), g(n) > M_n$*, for all* $n \in \mathbb{N}$*. Then* $(T_f, r) \cong (T_g, r)$ *if, and only if,* $f = g$*.*

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Since there are $2^{\aleph_0}$ $f, g \in \mathbb{N}^{\mathbb{N}}$ with $f(n), g(n) > M_n$ the result follows.

# THANKS!


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- Bruno, J. and Szeptycki, P. *There are exactly $\omega_1$ topological types of locally finite trees with countably many rays*, Fundamenta Mathematicae 256, 243-259, 2022.
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- Bruno, J. and Szeptycki, P. *Graph relations and TAC*, in preparation (2022).