Characterization of (semi-)Eberlein compacta using retractional skeletons

C. Correa, M. Cúth, J. Somaglia

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One of the main outcomes of our paper: class "semi-Eberlein + Corson" is stable under continous images.

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Summary: the crucial property of Valdivia compacta for functional analysis is the existence of “good-enough” system of retractions.

Let $K \subset [-1,1]^I$ be such that $K \cap \Sigma(I)$ is dense. Pick $A \in [I]^\omega$. Then

- inductively we construct $M(A) \in [I]^\omega$ with $A \subset M(A)$ such that $x|_{M(A)} \in K$ for every $x \in K$

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- $r_{M(B)}[K]$ is Valdivia and its weight is less or equal to $|B|$.

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Theorem (Bandlow, 1991 or maybe Kubis, 2009): K is Corson iff there exists a full retractional skeleton on K.

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$$|r_{A_n}x(i) - r_A x(i)| = \begin{cases} 0 & i \in M(A_n) \cup I \setminus M(A) \\ |x(i)| & i \in M(A) \setminus M(A_n). \end{cases}$$

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Summary: For every $x \in K$ we obtain $\lim_{n\to\infty} \sup_{i\in I} |r_{A_n}x(i) - r_A x(i)| = 0$.

Definition: Given $\mathcal{A} \subset \mathcal{C}(K)$ we say a retractional skeleton $(r_s)_{s\in\Gamma}$ is $\mathcal{A}$-shrinking if for every $x \in K$ and every increasing sequence (s_n) in Γ we have

$$\lim_{n\to\infty} \sup_{f\in\mathcal{A}} \left| f\left(r_{s_n}x\right) - f\left(r_s x\right) \right| = 0.$$

Theorem

Let K be a compact space. Then the following conditions are equivalent:

- *K is Eberlein.*
- *There exist a bounded set $\mathcal{A} \subset \mathcal{C}(K)$ separating the points of K and a retractional skeleton $\mathfrak{s} = (r_s)_{s\in\Gamma}$ on K such that $\mathfrak{s}$ is $\mathcal{A}$-shrinking.*

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Fabián, Montesinos, 2018 (reformulation): A compact space K is Eberlein if there exists a retractional skeleton (r_s) on K and $\mathcal{A} \subset \mathcal{C}(K)$ bounded and linearly dense such that for every $\mu \in \mathcal{C}(K)^*$ and every increasing sequence (s_n) in Γ we have

$$\lim_{n\to\infty} \sup_{f\in\mathcal{A}} \left|\mu(f \circ r_{s_n}) - \mu(f \circ r_s)\right| = 0.$$

Analogy for semi-Eberlein compacta gives the following.

Theorem

Let K be a compact space. Then the following conditions are equivalent:

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    - *$\mathfrak{s}$ is $\mathcal{A}$-shrinking with respect to D,*
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$$\varphi^* \mathcal{C}(L) = \{f \circ \varphi \colon f \in \mathcal{C}(L)\} \subset \mathcal{C}(K).$$

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- *the set $\mathcal{A}$ in $\mathcal{C}(L)$:* what is the set $\mathcal{A}$ which would work for $\varphi^*\mathcal{C}(L)$? (it should separate points of L, but we might have $\mathcal{A} \cap \varphi^*\mathcal{C}(L) = \emptyset$)

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- More concretely

$$\mathcal{A}_n := \left\{ \sum_{i=1}^{k} a_i \Pi_{j=1}^{n} f_{i,j} \colon f_{i,j} \in \mathcal{A},\ k \in \mathbb{N}\ , \sum_{i=1}^{k} |a_i| \leq n \right\}$$

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then $B_{\mathcal{C}(L)} = \bigcup_{n,m} \mathcal{B}_{n,m}$ for every $m \in \mathbb{N}$ and the properties of $\mathcal{A}_{n,m}$ related to the skeleton (r_M) should be transferred to the properties of $\mathcal{B}_{n,m}$ related to the skeleton (R_M) on L.

Theorem

Let K be a compact space. Then the following conditions are equivalent:

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Theorem

Let K be a compact space. Then the following conditions are equivalent:

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The strategy as mentioned above gives the following application:

Theorem

Let K be a compact space and $D \subset K$ be a dense subset such that there exists a homeomorphic embedding $h : K \to [-1,1]^J$ such that $h[D] = c_0(J) \cap h[K]$. Let us suppose that $\varphi : K \to L$ is a continuous surjection and $\varphi[D]$ is subset of the set induced by a retractional skeleton on L. Then there is a homeomorphic embedding $H : L \to [-1,1]^I$ with $H[\varphi[D]] \subset c_0(I)$. In particular, L is semi-Eberlein.

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Application - continuous images

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(3) If $\{(x,y) \in S \times S \colon \varphi(x) = \varphi(y)\}$ is dense in $\{(x,y) \in K \times K \colon \varphi(x) = \varphi(y)\}$, then condition (2) above is satisfied (Kalenda, 2000)

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Application 1: Obviously, if $S = K$ (which is iff K is Corson), then the condition above is satisfied. Thus, continuous image of Eberlein is Eberlein and continuous image of semi-Eberlein+Corson is semi-Eberlein+Corson.

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Application 2: If G_δ points are dense in L (or in K) and φ is open, then the condition above is satisfied as well (Kalenda, 2000).

Summary: K compact, $h: K \to [-1,1]^I$ embedding, $D := h^{-1}(c_0(I)) \subset K$ dense, $\varphi : K \to L$ continuous sujection. Put $S := h^{-1}(\Sigma(J))$. If

$$\overline{\{(x,y) \in S \times S \colon \varphi(x) = \varphi(y)\}} \supset \{(x,y) \in K \times K \colon \varphi(x) = \varphi(y)\}$$

then $\varphi(D) \subset L$ is dense and it embeds into $c_0(J)$ for some J. In particular, L is semi-Eberlein.

Application 1: Obviously, if $S = K$ (which is iff K is Corson), then the condition above is satisfied. Thus, continuous image of Eberlein is Eberlein and continuous image of semi-Eberlein+Corson is semi-Eberlein+Corson.

Application 2: If G_δ points are dense in L (or in K) and φ is open, then the condition above is satisfied as well (Kalenda, 2000). Thus, continuous open image of semi-Eberlein which has dense many G_δ points is semi-Eberlein (answers a Question posed by Kubis and Leiderman).

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THANK YOU FOR YOUR ATTENTION!