On the cardinality of a power homogeneous compactum

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We discuss the following theorem.

Main Theorem (C., 2021)

If X is a power homogeneous compactum then $|X| \leq 2^{at(X)\pi\chi(X)}$.

- A space X is *homogeneous* if for every $x, y \in X$ there exists a homeomorphism $h : X \to X$ such that $h(x) = y$.
- A space X is *power homogeneous* if there exists a cardinal κ such that X^κ is homogeneous.
- A *compactum* is a compact, Hausdorff space.
- The Hilbert Cube $[0,1]^\omega$ is homogeneous (Keller, 1931), thus $[0,1]$ is a power homogeneous compactum that is not homogeneous.
- $(\omega+1)^\omega$ is also homogeneous (van Douwen?), thus $\omega + 1$ is another example of a power homogeneous compactum that is not homogeneous.
- $at(X)$ satisfies $wt(X) \leq at(X) \leq t(X)$

Background

Theorem (Arhangel′skiĭ, 1970)

If X is a sequential homogeneous compactum then $|X| \leq \mathfrak{c}$.

Arhangel′skiĭ asked if “sequential” can be replaced with "countably tight”. R. de la Vega answered this in the affirmative.

Theorem (de la Vega, 2006)

If X is a homogeneous compactum then $|X| \leq 2^{t(X)}$.

de la Vega’s original proof involved the following critical theorem:

Theorem (Arhangel′skiĭ, 1978)

If X is a compactum and $t(X) \leq \kappa$ then there exists a non-empty closed set $G \subseteq X$ and a set $H \in [X]^{\leq\kappa}$ such that $\chi(G, X) \leq \kappa$ and $G \subseteq \overline{H}$.

A short proof of de la Vega's Theorem due to C. and Ridderbos appeared in 2011, using a result of Pytkeev.

Theorem (Arhangel′skiĭ, van Mill, Ridderbos, 2007)

If X is a power homogeneous compactum, then $|X| \leq 2^{t(X)}$.

Their proof involved the previous 1978 result of Arhangel′skiĭand the following technical result involving power homogeneity:

Theorem (AVR, 2007)

Let X be a power homogeneous Hausdorff space and suppose that $\pi\chi(X) \leq \kappa$ for a cardinal κ. Suppose there exists a nonempty G_κ-set G and a set $H \in [X]^{\leq\kappa}$ such that $G \subseteq \overline{H}$. Then there exists a cover $\mathcal{G}$ of X consisting of G^κ-sets such that for all $G \in \mathcal{G}$ there exists $H_G \in [X]^{\leq\kappa}$ such that $G \subseteq \overline{H_G}$.

Weak tightness

Definition (C., 2018)

Let X be a space. The *weak tightness* $wt(X)$ of X is defined as the least infinite cardinal κ for which there is a cover $\mathcal{C}$ of X such that $|\mathcal{C}| \leq 2^\kappa$ and for all $C \in \mathcal{C}$, $t(C) \leq \kappa$ and $X = cl_{2^\kappa} C$. We say that X is *weakly countably tight* if $wt(X) = \omega$.

Definition

Given a cardinal κ, a space X, and $A \subseteq X$, the κ-*closure* of A is defined as $cl_\kappa A = \bigcup_{B \in [A]^{\leq \kappa}} \overline{B}$.

It is clear that $wt(X) \leq t(X)$.

The weak tightness encodes the essential properties of tightness that prove sufficient to replace $t(X)$ with $wt(X)$ in certain cardinal inequalities.

Theorem (C., 2018)

If X is Hausdorff then $|X| \leq 2^{L(X)wt(X)\psi(X)}$.

Definition (Juhász, van Mill, 2018)

Given a cover $\mathcal{C}$ of X, a subset $A \subseteq X$ is $\mathcal{C}$-*saturated* if $A \cap C$ is dense in A for every $C \in \mathcal{C}$.

Proposition

Let X be a space, κ a cardinal such that $wt(X) \leq \kappa$, and $\mathcal{C}$ be a cover of X witnessing that $wt(X) \leq \kappa$. If $\mathcal{B}$ is an increasing chain of κ^+-many $\mathcal{C}$-saturated subsets of X, then

$$\overline{\bigcup \mathcal{B}} = \bigcup_{B \in \mathcal{B}} \overline{B}.$$

Theorem (Bella, C., 2019)

If X is a homogeneous compactum then $w(X) \leq 2^{wt(X)}$.

Using this result and the fact that $|X| \leq d(X)^{\pi\chi(X)}$ for homogeneous Hausdorff spaces, we have:

Theorem (Bella, C.)

If X is a homogeneous compactum then $|X| \leq 2^{wt(X)\pi\chi(X)}$.

This gives a general improvement of de la Vega's Theorem, as $\pi\chi(X) \leq t(X)$ for a compactum X and $wt(X) \leq t(X)$ for any space.

Question (Bella, C., 2019)

If X is a power homogeneous compactum, is $|X| \leq 2^{wt(X)\pi\chi(X)}$?

Main Theorem

Theorem

If X is a power homogeneous compactum then $|X| \leq 2^{at(X)\pi\chi(X)}$.

Definition (C., 2021)

Let X be a space. The *almost tightness* $at(X)$ of X is defined as the least infinite cardinal κ for which there is a cover $\mathcal{C}$ of X such that $|\mathcal{C}| \leq \kappa$ and for all $C \in \mathcal{C}$, $t(C) \leq \kappa$ and $X = cl_\kappa C$. We say that X is *almost countably tight* if $at(X) = \omega$.

Compare with the definition of weak tightness we saw earlier:

Definition

Let X be a space. The *weak tightness* $wt(X)$ of X is defined as the least infinite cardinal κ for which there is a cover $\mathcal{C}$ of X such that $|\mathcal{C}| \leq 2^\kappa$ and for all $C \in \mathcal{C}$, $t(C) \leq \kappa$ and $X = cl_{2^\kappa} C$. We say that X is *weakly countably tight* if $wt(X) = \omega$.

- It is clear that $wt(X) \leq at(X) \leq t(X)$.
- There are compact examples for which $at(X) < t(X)$, due to Spadaro and Szeptycki.

Proof Components

A G^c_κ-set is a set G for which there exists a family of open sets $\mathcal{U}$ such that $|\mathcal{U}| \leq \kappa$ and $G = \bigcap \mathcal{U} = \bigcap_{U \in \mathcal{U}} \overline{U}$.

Theorem

Let X be a power homogeneous Hausdorff space and suppose that $\pi\chi(X) \leq \kappa$ for a cardinal κ. Suppose there exists a nonempty G^c_κ-set G and a set $H \in [X]^{\leq\kappa}$ such that $G \subseteq \overline{H}$. Then there exists a cover $\mathcal{G}$ of X consisting of G^c_κ-sets such that for all $G \in \mathcal{G}$ there exists $H_G \in [X]^{\leq\kappa}$ such that $G \subseteq \overline{H_G}$.

If "$H \in [X]^{\leq\kappa}$" and "$H_G \in [X]^{\leq\kappa}$" in the above could be replaced with "$H \in [X]^{\leq 2^\kappa}$" and "$H_G \in [X]^{\leq 2^\kappa}$", respectively, then it could be shown that if X is a power homogeneous compactum then $|X| \leq 2^{wt(X)\pi\chi(X)}$, answering the question of Bella and C.

Proposition

Let X be a space, $at(X) = \kappa$, and let $\mathcal{C}$ be a cover witnessing that $at(X) = \kappa$. Then for all $x \in X$ there exists $T(x) \in [X]^{\leq \kappa}$ such that $x \in T(x)$ and $T(x)$ is $\mathcal{C}$-saturated.

Whenever $at(X) = \kappa$ and $x \in X$, we fix $T(x)$ as obtained in the above Proposition. If $A \subseteq X$, then we set $T(A) = \bigcup_{x \in A} T(x)$.

We introduce the notion of a T-free sequence for use with the invariant $at(X)$.

Definition (C., 2021)

Let $at(X) = \kappa$. A set $\{x_\alpha : \alpha < \lambda\}$ is an *T-free sequence* if $\overline{T(\{x_\beta : \beta < \alpha\})} \cap \overline{\{x_\beta : \alpha \leq \beta < \lambda\}} = \varnothing$ for all $\alpha < \lambda$.

Proposition

Let X be a space such that $at(X) = \kappa$. A compact subset $K \subseteq X$ contains no T-free sequence of length κ^+.

Theorem (C., 2021)

Let X be a Hausdorff space, $\kappa = at(X)$, and K a nonempty compact subset of X. Then there exists a nonempty closed set $G \subseteq K$ and a set $H \subseteq X$ such that $|H| \leq \kappa$, $G \subseteq \overline{H}$, and $\chi(G, K) \leq \kappa$. In addition, H is $\mathcal{C}$-saturated in any cover $\mathcal{C}$ witnessing that $at(X) = \kappa$.

This is an improvement over Arhangel′skiĭ's 1978 result.

Theorem (Arhangel′skiĭ, 1978)

If X is a compactum and $t(X) \leq \kappa$ then there exists a non-empty closed set $G \subseteq X$ and a set $H \in [X]^{\leq\kappa}$ such that $\chi(G, X) \leq \kappa$ and $G \subseteq \overline{H}$.

Two more components of the proof of the Main Theorem are needed.

Given a space X, X_κ^c represents the G_κ^c-*modification* of X, the space formed on X where the G_κ^c-sets form a basis.

Theorem (C., 2018)

For any space X and cardinal κ, $L(X_\kappa^c) \leq 2^{L(X)wt(X)\cdot\kappa}$.

Theorem (Ridderbos, 2006)

If X is a power homogeneous Hausdorff space then $|X| \leq d(X)^{\pi\chi(X)}$.

After putting these components together, we can prove:

Theorem

If X is a power homogeneous compactum then $|X| \leq 2^{at(X)\pi\chi(X)}$.

Corollary (Juhász, van Mill, 2018 (homogeneous case), C., 2018)

Let X be a power homogeneous compactum and suppose there exists a countable cover of X consisting of dense, countably tight subspaces. Then $|X| \leq \mathfrak{c}$.

The main theorem has a generalization to the Hausdorff setting.

Theorem

Let X be a power homogeneous Hausdorff space. Then $|X| \leq 2^{L(X)at(X)\pi\chi(X)pct(X)}$.

Further work

The following was introduced by Tkachenko in 1983:

Definition

The *o-tightness* of a space X does not exceed κ, or $ot(X) \leq \kappa$, if for every family $\mathcal{U}$ of open sets of X and for every point $x \in X$ with $x \in \overline{\bigcup \mathcal{U}}$ there exists a subfamily $\mathcal{V} \subseteq \mathcal{U}$ such that $|\mathcal{V}| \leq \kappa$ and $x \in \overline{\bigcup \mathcal{V}}$.

- It is clear that $ot(X) \leq t(X)$.
- It can also be shown that $ot(X) \leq c(X)$.
- What surprised me was this: $ot(X) \leq wt(X)$.
- Thus, $ot(X) \leq wt(X) \leq at(X) \leq t(X)$.

In light of previous results, we ask:

Question

If X is a homogeneous compactum, is $|X| \leq 2^{ot(X)\pi\chi(X)}$?

If the answer to the above is 'yes', it would simultaneously

1. improve the result that $|X| \leq 2^{wt(X)\pi\chi(X)}$ for a homogeneous compactum X, as $ot(X) \leq wt(X)$, and
2. generalize, in the compact case, the result that $|X| \leq 2^{c(X)\pi\chi(X)}$ for any Hausdorff, homogeneous space (C., Ridderbos, 2008), as $ot(X) \leq c(X)$.

Question (de la Vega)

If X is a homogeneous compactum, is $|X| \leq 2^{\pi\chi(X)}$?

Question (Bella, C.)

If X is a power homogeneous compactum, is $|X| \leq 2^{wt(X)\pi\chi(X)}$?

Question (Spadaro and Szeptycki)

Is there a (power) homogeneous compactum X such that $at(X) < t(X)$?

N. Carlson, *Power homogeneous compacta and variations on tightness*, to appear in Topology Appl.

Thank you!

Any questions?

Any answers?