# Generalized Ważewski Dendrites As Projective Fraïssé Limits

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# Projective Fraïssé Limits

A class $\mathcal{F}$ of finite topological $L$-structures and epimorphisms is called a **projective Fraïssé class** if:

- it contains only countably many structures up to isomorphism,
- its epimorphisms are closed under composition and contain the identity,
- for all $A, B \in \mathcal{F}$ there is $C \in \mathcal{F}$ with epimorphisms $C \to A$ and $C \to B$,
- for all diagrams of the form $A \xrightarrow{f} B \xleftarrow{g} C$ in $\mathcal{F}$ there exists $D \in \mathcal{F}$ with epimorphisms $h_1 : D \to B$ and $h_2 : D \to C$ such that $f \circ h_1 = g \circ h_2$.

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**Theorem (Irwin – Solecki, 2006; Panagiotopoulos – Solecki, 2018)**

*If $\mathcal{F}$ is a projective Fraïssé class, then there exists a unique $\mathbb{F} \in \mathcal{F}^\omega$, called its* ***projective Fraïssé limit****, such that:*

- *For all $A \in \mathcal{F}$ there exists an epimorphism $\mathbb{F} \to A$,*
- *If $f \colon \mathbb{F} \to A$ and $g \colon B \to A$ are epimorphisms with $A, B \in \mathcal{F}$, then there exists an epimorphism $h \colon \mathbb{F} \to B$ such that $g \circ h = f$.*

# Generalized Ważewski Dendrites

A **dendrite** is a continuum (compact, connected and metrizable topological space) which is uniquely arcwise connected and locally connected.

If $X$ is a dendrite and $x \in X$ we say that $x$ is an **endpoint** if $X \setminus \{x\}$ is connected, $x$ is a **regular point** if $X \setminus \{x\}$ has two connected components and $x$ is a **ramification point** if $X \setminus \{x\}$ has at least three connected components. In the latter case the **order** of $x$, denoted by $\mathrm{ord}(x)$ is the (possibly countably infinite) number of connected components of $X \setminus \{x\}$.

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### Theorem (Charatonik – Dilks, 1994)

*Given $P \subseteq \{3, 4, \ldots, \omega\}$ there is a unique dendrite, called the* ***generalized Ważewski dendrite*** *$W_P$, such that*

- *if $x \in W_P$ is a ramification point, then $\mathrm{ord}(x) \in P$,*
- *for every $p \in P$, the set of ramification points of order $p$ is arcwise dense in $W_P$.*

# Topological Graphs and Monotone Maps

A **topological graph** is a compact, zero-dimensional metrizable space $X$ equipped with a closed relation $E \subseteq X^2$ which is symmetric and reflexive. If additionally $E$ is transitive, $X$ is called a **prespace** and $|X| = X/E$ is called its **topological realization**.

A topological graph $X$ is **disconnected** if it is possible to write $X = A \sqcup B$ with $A, B$ clopen and such that there are no edges between $A$ and $B$. It is **connected** if it is not disconnected.

A continuous surjection $f\colon B \to A$ between topological graphs is called **monotone** if $f^{-1}(C)$ is connected in $B$ whenever $C \subseteq A$ is connected.

We have the following results:

**Theorem (Charatonik – Roe, 2021)**

*Let $\mathcal{F}$ be a projective Fraïssé class of trees with monotone epimorphisms, if its projective Fraïssé limit is a prespace, then it has a dendrite as topological realization.*

**Theorem (Charatonik – Roe, 2021)**

*The class of finite trees with monotone maps is a projective Fraïssé class. Its projective Fraïssé limit is a prespace whose topological realization is the Ważewski dendrite $W_3$.*

# Coherence

A monotone map $f : B \longrightarrow A$ between finite trees is called **weakly coherent** at a ramification point $a \in A$ if there exists a **witness** $b \in B$ such that the connected components of $A \setminus \{a\}$ can be enumerated as $\{A_1, \ldots, A_n\}$ and those of $B \setminus \{b\}$ as $\{B_1, \ldots, B_m\}$ with $m \geq n$ and $f^{-1}(A_i) \subseteq B_i$ for every $1 \leq i \leq n$.

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Fix $\mathcal{F}$ a projective Fraïssé family of trees with monotone maps. Let $\langle F_i \mid i < \omega \rangle$ with maps $f_i^j\colon F_j \to F_i$ for $j \geq i$ be a projective Fraïssé sequence for $\mathcal{F}$. Its projective Fraïssé limit $\mathbb{F}$ can be identified with a subspace of $\prod F_i$.

A point $x = (x_i)_{i<\omega} \in \mathbb{F}$ is called **(weakly) coherent** if there exists $N \in \mathbb{N}$ such that for all $i > N$, $x_{i+1}$ witnesses the (weak) coherence of $f_i^{i+1}$ at $x_i$.

### Theorem (C. – Kwiatkowska, 2022)

*Let $\mathcal{F}$ be a projective Fraïssé family of finite trees with coherent maps and suppose that its projective Fraïssé limit $\mathbb{F}$ is a prespace. Then the topological realization $\pi\colon \mathbb{F} \to |\mathbb{F}|$ is a bijection between ramification points of $|\mathbb{F}|$ and weakly coherent points of $\mathbb{F}$.*

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## Remark

To prove that $x \in \mathbb{F}$ is weakly coherent implies that $\pi(x)$ is a ramification point it's enough for the morphisms in $\mathcal{F}$ to be monotone. To prove the reverse implication they must be coherent.

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To prove that $x \in \mathbb{F}$ is weakly coherent implies that $\pi(x)$ is a ramification point it's enough for the morphisms in $\mathcal{F}$ to be monotone. To prove the reverse implication they must be coherent.

If $x = (x_i)_{i<\omega} \in \mathbb{F}$ is weakly coherent either $\mathrm{ord}(x_i)$ stabilizes to some finite value $n$, in which case $x$ is a coherent point and $\mathrm{ord}(\pi(x)) = n$, or $\mathrm{ord}(x_i) \to \infty$, in which case $\mathrm{ord}(\pi(x)) = \omega$.

# Construction of the Projective Fraïssé Families

Fix $P \subseteq \{3, 4, \ldots, \omega\}$. We introduce two families of finite topological graphs, depending on whether $\omega \in P$ or not.

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**Case 1:** $\omega \notin P$. Let $\mathcal{G}_P$ be the family of finite trees with no vertices of order two, and such that every vertex is either an endpoint or has order in $P$. A map $f : B \to A$ is an epimorphism in $\mathcal{G}_P$ iff it is coherent.

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## Theorem (C. – Kwiatkowska, 2022)

*The family $\mathcal{G}_P$ is a projective Fraïssé family. Its projective Fraïssé limit is a prespace whose topological realization is the generalized Ważewski dendrite $W_P$.*

**Case 2:** $\omega \in P$. Let $\mathcal{F}_P$ be the family of finite trees with no vertices of order two. A map $f : B \longrightarrow A$ between trees in $\mathcal{F}_P$ is an epimorphism in $\mathcal{F}_P$ iff

- it is monotone,
- for all $a \in A$ with $\operatorname{ord}(a) \in P$, $f$ is coherent at $a$,
- for all $a \in A$ with $\operatorname{ord}(a) \notin P$, $f$ is weakly coherent at $a$, and $\operatorname{ord} b \notin P$, where $b \in B$ is the witness for the weak coherence of $f$ at $a$.

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### Theorem (C. – Kwiatkowska, 2022)

*The family $\mathcal{F}_P$ is a projective Fraïssé family whose projective Fraïssé limit $\mathbb{F}_P$ is a prespace. If $P$ is coinfinite then $|\mathbb{F}| \cong W_P$, otherwise $|\mathbb{F}_P| \cong W_{P'}$, where $P' = P \setminus \{\omega\} \cup \max\{a \notin P \mid \forall n > a \, (n \in P)\}$.*

# Removing the Coinfiniteness Assumption

Let $P \subseteq \{3, 4, \ldots, \omega\}$ and consider the language $L_P = \{R\} \cup \{U_p \mid p \in P\}$, where every $U_p$ is a unary predicate. We construct $\mathcal{F}_P$ as follows:

- a structure in $\mathcal{F}_P$ is a finite tree with at least one ramification point and every ramification point $x$ is labelled with exactly one $U_p$ such that $\text{ord}(x) \leq p$ and endpoints are not labelled;

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- a structure in $\mathcal{F}_P$ is a finite tree with at least one ramification point and every ramification point $x$ is labelled with exactly one $U_p$ such that $\mathrm{ord}(x) \leq p$ and endpoints are not labelled;
- an epimorphism $f \colon B \longrightarrow A$ in $\mathcal{F}_P$ is given by a pair of maps $p(f) \colon B \longrightarrow A$ and $e(f) \colon \mathrm{End}(A) \rightarrow \mathrm{End}(B)$ such that
  - $p(f) \colon B \longrightarrow A$ is weakly coherent and if $U_p(a)$ holds for a ramification point $a \in A$, then $U_p(b)$ holds for the witness of the coherence of $p(f)$ at $a$;
  - $e(f)$ is an injection on endpoints such that $p(f) \circ e(f) = \mathrm{Id}_{\mathrm{End}(A)}$.

The limit is now a pair $(\mathbb{F}_P, E)$: $\mathbb{F}_P$ is a topological graph obtained as the inverse limit of the projection part, while $E \subseteq \mathrm{End}(\mathbb{F}_P)$ is a countable dense subset obtained as the direct limit of the embedding part.

### Theorem (C. – Kwiatkowska, 2022)

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Since projective Fraïssé limits are unique up to homeomorphism we recover the following countable dense homogeneity result.

**Theorem (Charatonik – Dilks, 1994)**

*Let $E, F \subseteq \mathrm{End}(W_P)$ be two countable dense subsets. Then there exists $h \in \mathrm{Homeo}(W_P)$ such that $h(E) = F$.*

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