On ”differently” characterized subgroups of the circle group

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Throughout $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{Z}$ and ω will stand for the set of all real numbers, the set of all rational numbers, the set of all integers and the set of all natural numbers respectively. The first three are equipped with their usual abelian group structure and the circle group $\mathbb{T}$ is identified with the quotient group $\mathbb{R}/\mathcal{Z}$ of $\mathbb{R}$ endowed with its usual compact topology. For $x \in \mathbb{R}$ we denote by $\{x\}$ the difference $x - [x]$ (the fractional part) and $\|x\|$ the distance from the integers i.e. $\min\{\{x\}, 1 - \{x\}\}$.

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- The motivation to study the so called ”characterized subgroups” can be traced back to the distribution of sequences of multiples of a given real number mod 1.
- Recall that a sequence of real numbers (x_n) is said to be uniformly distributed mod 1, if for every $[a, b] \subseteq [0, 1)$ one has

$$\lim_{n\to\infty} \frac{|\{j : 0 \le j < n,\ \{x_j\} \in [a, b]\}|}{n} = b - a$$

where $\{x_j\}$ is the fractional part of x_j. In his celebrated results proved in 1916, H. Weyl had investigated the set

$$W_{\mathbf{u}} = \{x \in [0, 1] : (u_n x) \text{ is uniformly distributed mod } 1\}$$

where $\mathbf{u} = (u_n) \in \mathbb{Z}^\omega$.

- Note that for every number $\alpha \in [0,1] \setminus \mathbb{Q}$, $\alpha \notin W_{\mathbf{u}}$ for an appropriate choice of **u**. Indeed, to this end one can consider the convergents $\frac{r_n}{u_n}$ of the continued fraction expansion of α and as $||u_n\alpha||_{\mathbb{Z}} \to 0$ (where $||.||_{\mathbb{Z}}$ is the distance from the integers), conclude that $\alpha \notin W_{\mathbf{u}}$.

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- In a really impressive observation, [Larcher, PAMS, 1988] proved that if the continued fraction expansion of $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ is bounded then

$$\{\beta \in \mathbb{R} : ||u_n\beta||_{\mathbb{Z}} \to 0\} = \langle\alpha\rangle + \mathbb{Z}, \tag{1}$$

the subgroup of $\mathbb{R}$ generated by α mudulo 1. Instead of using the fractional part $\{x_j\}$ or working modulo 1, one can conveniently work in the circle group $\mathbb{R} \setminus \mathbb{Z} = \mathbb{T}$

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- It is obvious that any p-torsion element is topologically p-torsion.
- [Armacost, 1981] defined the subgroups

$$X_p = \{x \in X : p^n x \to 0\} \text{ and } X! = \{x \in X : n!x \to 0\}$$

of an abelian topological group X, and started their investigation

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$$t_{(a_n)}(\mathbb{T}) := \{x \in \mathbb{T} : a_n x \to 0 \text{ in } \mathbb{T}\}$$

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- Even if the notion was inspired by the various (earlier) instances, the term *characterized* appeared much later, coined in [Bíró, Deshouillers, Sós, SSMH, 2001].

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Example

(a) Let p be a prime. For the sequence (a_n), defined by $a_n = p^n$ for every n, obviously $t_{(p^n)}(\mathbb{T})$ contains the Prüfer group $\mathcal{Z}(p^\infty)$. Armacost proved that $t_{(p^n)}(\mathbb{T})$ simply coincides with $\mathcal{Z}(p^\infty)$.
(b) Armacost posed the problem to describe the group $\mathbb{T}! = t_{(n!)}(\mathbb{T})$. It was resolved by [Borel, CM, 1991].

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- Precisely a sequence of positive integers (a_n) is an arithmetic sequence if

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- If an irrational number α has the regular continued fraction approximation $\alpha = [a_0; a_1, a_2, ...]$, For any $n \in \mathbb{N}$, let $\frac{p_n}{q_n} = [a_0; a_1, a_2, ..., a_n]$ be the sequence of convergents and we write $\theta_n = q_n\alpha - p_n$. There have been a lot of interest in the characterized subgroup generated by the sequence of denominators (q_n) i.e. the subgroup $t_{(q_n)}(\mathbb{T})$.

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- [Eggleston, PLMS, 1952] had observed that when (a_n) is an arithmetic sequence:
  - (E1) $t_{(a_n)}(\mathbb{T})$ is countable if $(\frac{a_n}{a_{n-1}})$ is bounded,
  - (E2) $|t_{(a_n)}(\mathbb{T})| = 2^{\aleph_0}$ if $(\frac{a_n}{a_{n-1}}) \to \infty$.

- For an irrational number $\alpha = [a_0; a_1, a_2, ...]$, [Kraaikamp, Liardet, PAMS, 1991]

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- [Bíró, Deshouillers and Sós, SSMH, 2001] established the important fact that every countable subgroup of $\mathbb{T}$ is characterized.
- The whole history concerning these investigations along with relevant references can be seen from the excellent survey article on characterized subgroups of $\mathbb{T}$ [Di Santo, Dikranjan, Giordano Bruno, Ric. Mat, 2018]).

- **Definition 2.** [Buck, AJM, 1946] By $|A|$ we denote the cardinality of a set A. The lower and the upper natural densities of $A \subset \omega$ are defined by

$$\underline{d}(A) = \liminf_{n\to\infty} \frac{|A \cap [1,n]|}{n} \quad \text{and} \quad \overline{d}(A) = \limsup_{n\to\infty} \frac{|A \cap [1,n]|}{n}.$$

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- **Observation:** We say that a subset of ω is "small" if it has natural density zero. We write $\mathcal{I}_d = \{A \subset \omega : d(A) = 0\}$. Evidently $\mathcal{I}_d$ forms an ideal (i.e. $\omega \notin \mathcal{I}_d$, it is hereditary and closed under finite unions).

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- **Definition 3.** [Fast, CM, 1951,Steinhaus, CM, 1951] A sequence (x_n) in (X, ρ) is said to be statistically convergent to $x_0 \in X$ if for arbitrary $\varepsilon > 0$ the set $K(\varepsilon) = \{n \in \omega : \rho(x_n, x_0) \geq \varepsilon\}$ has natural density zero.

- [Salat, MS, 1980] A sequence (x_n) of real numbers is statistically convergent to ξ if and only if there exist a set $M = \{m_1 < m_2 < ...\} \subset \omega$ such that $d(M) = 1$ and $\lim_{k\to\infty} x_{m_k} = \xi$.

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- $\star$ This particular property of statistical convergence make it "non-trivial" yet "not too wild" and this is the reason why it has been used to extend several classical results and present new characterizations of existing concepts.
- A metric space (X, d) is complete iff every statistically cauchy sequence is statistically convergent in X.
- the class of functions obtained as statistical limits of sequences of continuous functions coincides with the usual "Baire class one functions" in a metric space (X, d).

Definition (Dikranjan, Das, Bose, FM, 2020)

For a sequence of integers (a_n) the subgroup

$$t^s_{(a_n)}(\mathbb{T}) := \{x \in \mathbb{T} : a_n x \to 0 \text{ statistically in } \mathbb{T}\}$$

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⋆ Even if the correspondence $(a_n) \mapsto t_{(a_n)}(\mathbb{T})$ is monotone decreasing (with respect to inclusion), in many cases (as in the classical examples) the subgroup $t_{(a_n)}(\mathbb{T})$ is rather small, even if the sequence (a_n) is not too dense (in the above example, it is a geometric progression, so has exponential growth). This suggests that asking $a_n x \to 0$ is maybe somewhat too restrictive.

Theorem

For any sequence of integers (a_n), $t^s_{(a_n)}(\mathbb{T})$ is a $F_{\sigma\delta}$ (hence, Borel) subgroup of $\mathbb{T}$ containing $t_{(a_n)}(\mathbb{T})$.

- In general the subgroup $t^s_{(a_n)}(\mathbb{T})$ may not be complete with respect to the usual norm $\|.\|$ prevalent in $\mathbb{T}$
- Let $\delta : \mathbb{T} \times \mathbb{T} \to \mathbb{R}$ be defined as follows. For any $x, y \in \mathbb{T}$, let

$$\delta(x, y) = \sup_{n\in\mathbb{N}}\{\|x - y\|, \|a_n(x - y)\|\}.$$

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Corollary

There is a finer topology on the subgroup $t^s_{(a_n)}(\mathbb{T})$ *which is completely metrizable.*

Lemma (see Dikranjan, Impieri, CA, 2014)

For any arithmetic sequence (a_n) and $x \in \mathbb{T}$, we can build a unique sequence of integers (c_n), where $0 \leq c_n < q_n$, such that

$$x = \sum_{n=1}^{\infty} \frac{c_n}{a_n} \tag{2}$$

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• For $x \in \mathbb{T}$ with canonical representation (2), we define $supp(x) = \{n \in \mathbb{N} : c_n \neq 0\}$ and $supp_q(x) = \{n \in \mathbb{N} : c_n = q_n - 1\}$. Clearly $supp_q(x) \subseteq supp(x)$.

- A typical example for the sequence (2^n): Choose $x \in \mathbb{T}$ with

$$supp_{(2^n)}(x) = \bigcup_{n=1}^{\infty} [(2n)^2, (2n+1)^2]. \tag{3}$$

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- the element $x \in \mathbb{T}$ above can be replaced by a more generally defined element of $\mathbb{T}$ by taking the support from $\mathbb{I}$ where
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$$\mathbb{I} = \{\bigcup_{n=1}^{\infty} B_n : B_n = [b_n, d_n], b_{n+1} > d_n + 1 \ \ \forall n; \}$$

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- $|\mathbb{I}| = \mathfrak{c}$.
- Fix a specific member $B = \bigcup_{n=1}^{\infty} B_n \in \mathbb{I}$. Fix a sequence $\xi = (z_i) \in \{0,1\}^{\mathbb{N}}$ and define $B^{\xi} = \bigcup_{k=1}^{\infty} B_{2k+z_k}$ (B^{ξ} of B is obtained by taking at each stage k either B_{2k} of B_{2k+1} depending on the choice imposed by ξ). Also $B^{\xi} \neq B^{\eta}$. for distinct $\xi, \eta \in \{0,1\}^{\mathbb{N}}$, which provides an injective map given by

  $$\{0,1\}^{\mathbb{N}} \ni \xi \to B^{\xi} \in \mathbb{I},$$

Theorem

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Theorem

For any arithmetic sequence (a_n), $t^s_{(a_n)}(\mathbb{T}) \neq t_{(a_n)}(\mathbb{T})$.

Lemma

[Das, Ghosh, BSM, 2022] Let (u_n) *be an arithmetic sequence and* (a_n) *be an increasing sequence of naturals. If* $G = \{\frac{1}{u_n} : n \in \mathbb{N}\} \subseteq t_{(a_n)}(\mathbb{T})$ *then* a_n *must be of the form* $u_{k_n} v_n$ *where* $k_n \to \infty$ *and* q_{k_n+1} *does not divide* v_n *for any* $n \in \mathbb{N}$.

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Theorem

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• It has been established that every member of the class of *s*-characterized subgroups is essentially new i.e. they can never be “characterized” by a sequence of integers.

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Let $\alpha \in (0,1)$ be an irrational number. Then the associated s-characterized subgroup $t^s_{(q_n)}(\mathbb{T})$ is uncountable.

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• Let (f_n) be the celebrated Fibonacci sequence, i.e. $f_1 = 1, f_2 = 1$ and for all $n \geq 3$, $f_n = f_{n-1} + f_{n-2}$. Then the associated s-characterized subgroup $t^s_{(f_n)}(\mathbb{T})$ is uncountable.

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• The family $\mathcal{I}_g = \{A \subset \omega : \overline{d}_g(A) = 0\}$ forms an ideal. It has been observed that $\omega \in \mathcal{I}_g$ iff. $\frac{n}{g(n)} \to 0$. So we additionally assume that $n/g(n) \nrightarrow 0$ so that $\omega \notin \mathcal{I}_g$ and it can be proved that $\mathcal{I}_g$ is a proper admissible P-ideal of ω. The collection of all such functions g satisfying the above mentioned properties will be denoted by G.

- An interesting observation [Das, Bose, PMH, 2021]

Theorem

Let $\alpha_1, \alpha_2 \in (0, 1]$ *with* $\alpha_1 < \alpha_2$. *Then* $t^{\alpha_1}_{(2^n)}(\mathbb{T}) \subsetneq t^{\alpha_2}_{(2^n)}(\mathbb{T})$.

Theorem

For $\alpha \in (0, 1]$, $|t^{\alpha}_{(2^n)}(\mathbb{T})| = \mathfrak{c}$.

Theorem

Both the set differences $\bigcap_{\alpha\in(0,1)} t^{\alpha}_{(2^n)}(\mathbb{T}) \setminus t_{(2^n)}(\mathbb{T})$ *and* $t^{s}_{(2^n)}(\mathbb{T}) \setminus \bigcup_{\alpha\in(0,1)} t^{\alpha}_{(2^n)}(\mathbb{T})$ *are non-empty.*

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THANK YOU FOR YOUR ATTENTION