Entropy of amenable monoid actions

Dikran Dikranjan
Udine University, Italy

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joint work with **Anna Giordano Bruno** and **Simone Virili** joint work with **Anna Giordano Bruno** and **Simone Virili**

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- for a topological space (X, τ), T is *continuous*; produces *topological entropy* $h_{top}(T)$.

- for an abelian group $(X, +)$, T is a *homomorphism*; produces *algebraic entropy* $h_{alg}(T)$.

In both cases we have a self-map $T : X \to X$ that defines a left action $\mathbb{N} \overset{\lambda}{\curvearrowright} X$ of the monoid $(\mathbb{N}, +)$ on X in the standard way $\lambda(n) = T^n$. Later the definition of entropy was extended to actions $S \overset{\lambda}{\curvearrowright} X$ of amenable monoids S on compact space X or a discrete group X (definitions follow).

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Let us call **Bridge Theorem** this remarkable equality.

Peters 1979 verified the Bridge Theorem for automorphisms of metrizable compact Abelian groups ($\mathbb{Z}$-actions). Giordano Bruno and DD [2010], verified the Bridge Theorem for all continuous endomorphisms of arbitrary compact Abelian groups ($\mathbb{N}$-actions).

This talk is dedicated to the Bridge Theorem and its applications.

Theorem (Bridge Theorem)

If S is a cancellative right amenable monoid, K a compact Abelian group and $K \overset{\rho}{\curvearrowleft} S$ a right S-action, then $h_{\rm top}(\rho) = h_{\rm alg}(\rho^\wedge)$.

Proved by H.Li [2012] for S a countable amenable group and K compact metrizable and some sofic group action by Liang [2019].

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Amenability and the left Ore condition

A *right Følner net* for a monoid S is a net $\{F_i\}_{i\in I}$ in $\mathcal{P}_{\rm fin}(S) = [S]^{<\omega} \setminus \{\emptyset\}$ such that $\lim_{i\in I} \frac{|F_i s \setminus F_i|}{|F_i|} = 0$ for every $s \in S$. We say that a cancellative monoid S is right amenable if it admits a right Følner net. (Amenability can be defined using finitely additive right invariant measures.)

Example

$(\mathbb{N}, +)$ is amenable, witnessed by the Følner sequence $F_n = \{0, 1, \ldots, n-1\}$. Every commutative monoid is amenable.

A cancellative monoid S is left Ore, if: for any pair of elements $s, t \in S$, the intersection $Ss \cap St \neq \emptyset$ is not trivial.
Clearly, S is left Ore iff $(S, \leq)$ is directed, with the partial preorder defined by $s \leq s'$ iff $s' = ts$ for some $t \in S$.
A cancellative and right amenable monoid S is always left Ore, and therefore, S can be embedded in a group $G := S^{-1}S$ that we call group of left fractions of S, then G is amenable.

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The category $\mathfrak{M}$ of normed monoids

An objects of $\mathfrak{M}$ is a *normed monoid*, i.e., a pair (M, v) where $(M, +)$ is a commutative monoid and $v\colon M \to \mathbb{R}_{\geq 0}$ is a function. A morphism $\phi\colon (M_1, v_1) \to (M_2, v_2)$ in $\mathfrak{M}$ is a contracting monoid homomorphism $\phi\colon M_1 \to M_2$ (i.e., $v_2(\phi(m)) \leq v_1(m)$ for all $m \in M_1$). So, ϕ is an *isomorphism* in $\mathfrak{M}$ if it is a monoid isomorphism and $v_2(\phi(m)) = v_1(m)$ for all $m \in M_1$.

The norm v of normed monoid (M, v) is said to be:

- *monotone* provided $v(x) \leq v(x + y)$, for all $x, y \in M$;
- *sub-additive* provided $v(x + y) \leq v(x) + v(y)$, for all $x, y \in M$.

The entropies h_{alg} and h_{top} are based on the following normed monoids (other entropies can be obtained using other normed monoids).

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Example (1)

Let X be a discrete Abelian group and $\mathfrak{F}(X)$ be the family of all finite symmetric subsets of X containing 0. The pair $(\mathfrak{F}(X), +)$ is a commutative monoid (as $F_1 + F_2 = F_2 + F_1$ for $F_1, F_2 \in \mathfrak{F}(X)$), with norm defined by $v_{\mathfrak{F}}(F) = \log |F|$, for all $F \in \mathfrak{F}(X)$. The norm $v_{\mathfrak{F}}$ is both monotone and sub-additive.

Example (2)

1. Let K be a compact space and $\mathrm{cov}(K)$ the family of its open covers. For $\mathcal{U}, \mathcal{V} \in \mathrm{cov}(K)$ let $\mathcal{U} \vee \mathcal{V} = \{U \cap V : U \in \mathcal{U}, V \in \mathcal{V}\}$. Then $(\mathrm{cov}(K), \vee)$ is a commutative monoid with a monotone and sub-additive norm given by $v_{\mathrm{cov}}(\mathcal{U}) = \log N(\mathcal{U})$ for all for all $\mathcal{U} \in \mathrm{cov}(K)$, where $N(\mathcal{U}) = \min\{|\mathcal{V}| : \mathrm{cov}(K) \ni \mathcal{V} \subseteq \mathcal{U}\}$.
2. Let K be a compact group, μ its Haar measure K and $\mathfrak{U}(K)$ be the family of all symmetric compact neighborhoods of 0 in K. Then the pair $(\mathfrak{U}(K), \cap)$ is a commutative monoid, with norm $v_{\mathfrak{U}}$ defined by $v_{\mathfrak{U}}(U) = -\log \mu(U)$, for each $U \in \mathfrak{U}(K)$. Clearly, $v_{\mathfrak{U}}$ is monotone, but not subadditive in general.

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1. Let K be a compact space and $\mathrm{cov}(K)$ the family of its open covers. For $\mathcal{U}, \mathcal{V} \in \mathrm{cov}(K)$ let $\mathcal{U} \vee \mathcal{V} = \{U \cap V : U \in \mathcal{U}, V \in \mathcal{V}\}$. Then $(\mathrm{cov}(K), \vee)$ is a commutative monoid with a monotone and sub-additive norm given by $v_{\mathrm{cov}}(\mathcal{U}) = \log N(\mathcal{U})$ for all for all $\mathcal{U} \in \mathrm{cov}(K)$, where $N(\mathcal{U}) = \min\{|\mathcal{V}| : \mathrm{cov}(K) \ni \mathcal{V} \subseteq \mathcal{U}\}$.
2. Let K be a compact group, μ its Haar measure K and $\mathfrak{U}(K)$ be the family of all symmetric compact neighborhoods of 0 in K. Then the pair $(\mathfrak{U}(K), \cap)$ is a commutative monoid, with norm $v_{\mathfrak{U}}$ defined by $v_{\mathfrak{U}}(U) = -\log \mu(U)$, for each $U \in \mathfrak{U}(K)$. Clearly, $v_{\mathfrak{U}}$ is monotone, but not subadditive in general.

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Actions and trajectories in $\mathfrak{M}$

Let G be a fixed infinite cancellative right amenable monoid and $M = ((M, +), v)$ a normed monoid. A G-action $G \overset{\alpha}{\curvearrowright} M$ on M is a monoid homomorphism $\alpha\colon G \to \mathrm{End}(M)$ (where $\mathrm{End}(M)$ is the monoid of all endomorphisms of normed monoids $M \to M$). For $x \in M$ and $F = \{f_1, \dots, f_k\} \subseteq G$, define the F-trajectory of x by

$$T_F(\alpha, x) = \alpha_{f_1}(x) + \ldots + \alpha_{f_k}(x).$$

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1. α_2 *dominates* α_1 if, for each $x \in M_1$, there exists $y \in M_2$ such that, $v_1(T_F(\alpha_1, x)) \leq v_2(T_F(\alpha_2, y))$ for all $F \in \mathcal{P}_{\mathrm{fin}}(G)$,

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Let $M = (M, v)$ be a normed monoid with v monotone, $G \overset{\alpha}{\curvearrowright} M$ a left G-action. Then for a right Følner net $\mathfrak{s} = \{F_i\}_{i \in I}$ of G the $\mathfrak{s}$-entropy of α at $m \in M$ is

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On the normed monoids in Examples (1) and (2), one has the following G-actions induced by a left G-action $G \overset{\lambda}{\curvearrowright} X$ and by a right G-action $K \overset{\rho}{\curvearrowleft} G$, respectively on a discrete Abelian group X and on a compact space K.

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Example (topological entropy)

Let K be a compact space and $K \overset{\rho}{\curvearrowleft} G$ a right G-action.

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For any $F \in \mathcal{P}_{\mathrm{fin}}(G)$, $\mathcal{U} \in \mathrm{cov}(K)$ and $U \in \mathfrak{U}(K)$,

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In particular, for any right Følner net $\mathfrak{s}$ for G, $H(\rho_{\mathrm{cov}}, \mathfrak{s}, \mathcal{U}) = H_{\mathrm{top}}(\rho, \mathcal{U})$ and $h(\rho, \mathfrak{s}) = h_{\mathrm{top}}(\rho)$ is the topological entropy [Ceccherini-Silberstein, M. Coornaert, F. Krieger 2014].

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Example (topological entropy)

Let K be a compact space and $K \overset{\rho}{\curvearrowleft} G$ a right G-action.
Define the left G-actions:

1. $G \overset{\rho_{\mathrm{cov}}}{\curvearrowright} \mathrm{cov}(K)$, by $(\rho_{\mathrm{cov}})_g(\mathcal{U}) = \rho_g^{-1}(\mathcal{U})$, for every $g \in G$;
2. $G \overset{\rho_{\mathfrak{U}}}{\curvearrowright} \mathfrak{U}(K)$, by $(\rho_{\mathfrak{U}})_g(U) = \rho_g^{-1}(U)$, for every $g \in G$.

For any $F \in \mathcal{P}_{\mathrm{fin}}(G)$, $\mathcal{U} \in \mathrm{cov}(K)$ and $U \in \mathfrak{U}(K)$,

$$T_F(\rho_{\mathrm{cov}}, \mathcal{U}) = \bigvee_{g \in F} \rho_g^{-1}(\mathcal{U}) \quad \text{and} \quad T_F(\rho_{\mathfrak{U}}, U) = \bigcap_{g \in F} \rho_g^{-1}(U)$$

In particular, for any right Følner net $\mathfrak{s}$ for G,
$H(\rho_{\mathrm{cov}}, \mathfrak{s}, \mathcal{U}) = H_{\mathrm{top}}(\rho, \mathcal{U})$ and $h(\rho, \mathfrak{s}) = h_{\mathrm{top}}(\rho)$ is the topological entropy [Ceccherini-Silberstein, M. Coornaert, F. Krieger 2014].

On the other hand, when K is a (locally) compact group, $h(\rho_{\mathfrak{U}}, \mathfrak{s})$ coincides with Bowen's entropy h_{Bowen} with respect to $\mathfrak{s}$.

Example (algebraic entropy)

Let X be a discrete Abelian group and $G \overset{\lambda}{\curvearrowright} X$ a left G-action.

The left G-action $G \overset{\lambda_{\mathfrak{F}}}{\curvearrowright} \mathfrak{F}(X)$ is defined by $(\lambda_{\mathfrak{F}})_g(F) = \lambda_g(F)$ for $g \in G, F \in \mathcal{P}_{\text{fin}}(G)$. Then for any $F \in \mathcal{P}_{\text{fin}}(G)$ and $E \in \mathfrak{F}(X)$,

$$T_F(\lambda_{\mathfrak{F}}, E) = \sum_{g \in F} \lambda_g(E),$$

is the $\lambda_{\mathfrak{F}}$-trajectory of L with respect to F.

The limit $H_{\text{alg}}(\lambda, E) := H(\lambda_{\mathfrak{F}}, \mathfrak{s}, E)$ (for some right Følner net $\mathfrak{s}$ for G) is the algebraic entropy of λ w.r.t. E and $h_{\text{alg}}(\lambda) := h(\lambda_{\mathfrak{F}}, \mathfrak{s})$ – the algebraic entropy of λ, as defined by Fornasiero, Giordano Bruno, DD [2019] (for $\mathbb{N}$-actions h_{alg} was introduced by Giordano Bruno, DD [2010], for $\mathbb{Z}$-actions it coincides with Peters' entropy h_{alg} although his definition cannot be extended to $\mathbb{N}$-actions).

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Invariance of the entropy under asymptotic equivalence

The following lemma plays a key role in the proof of the Bridge Theorem:

Lemma

Let $M_1 = (M_1, v_1)$ and $M_2 = (M_2, v_2)$ be two normed monoids, and $G \overset{\alpha_1}{\curvearrowright} M_1$, $G \overset{\alpha_2}{\curvearrowright} M_2$ left G-actions. If α_1 and α_2 are asymptotically equivalent, then $h(\alpha_1, \mathfrak{s}) = h(\alpha_2, \mathfrak{s})$ for every right Følner net $\mathfrak{s}$ for G.

Theorem

If G is an amenable group and $K \overset{\rho}{\curvearrowleft} G$ a right linear action on a compact group K, then $G \overset{\rho_{\mathfrak{U}}}{\curvearrowright} \mathfrak{U}(K)$ and $G \overset{\rho_{\mathrm{cov}}}{\curvearrowright} \mathrm{cov}(K)$ are equivalent. So, $h(\rho_{\mathfrak{U}}, \mathfrak{s}) = h(\rho_{\mathrm{cov}}, \mathfrak{s})$ for every Følner net $\mathfrak{s}$ for G.

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Adding "salt" from harmonic analysis

For an infinite LCA group Γ let $\mathfrak{U}(\Gamma)$ be the family of symmetric compact neighborhoods of $0 \in \Gamma$ and μ be a fixed Haar measure. Our main interest is in the case when $\Gamma = X$ is discrete (so μ is the counting measure) and when $\Gamma = K$ is compact (when there is a unique Haar measure such that $\mu(K) = 1$.

$L^1(\Gamma)$ – the space of absolutely integrable functions $\phi\colon \Gamma \to \mathbb{C}$ (those having $||\phi||_1 = \int_{x\in\Gamma} |\phi(x)|\delta\mu(x) < \infty$), identifying those that coincide almost everywhere, so that $||-||_1$ is a norm on $L^1(\Gamma)$.
$\mathfrak{P}(\Gamma)$ – the set of continuous and positive-definite functions on Γ ($\phi\colon \Gamma \to \mathbb{C}$, is positive-definite if $\sum_{i,j=1}^{n} c_i\overline{c_j}\phi(x_i - x_j) \in \mathbb{R}_{\geq 0}$, for all $n \in \mathbb{N}_{>0}$, $x_1, \ldots, x_n \in \Gamma$ and $c_1, \ldots, c_n \in \mathbb{C}$).

If $\phi, \psi \in L^1(\Gamma)$ then $\int_{y\in\Gamma} |\phi(y)\psi(x-y)|\delta\mu(y) < \infty$ for almost all $x \in \Gamma$, so the convolution

$$(\phi * \psi)(x) = \int_{y\in\Gamma} \phi(y)\psi(x-y)\delta\mu(y)$$

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Algebraic Peters normed monoids

Let $\mathcal{M}(\Gamma) = \{\phi \in L^1(\Gamma) \cap \mathfrak{P}(\Gamma) : \phi(\Gamma) \subseteq \mathbb{R}_{\geq 0}\} \setminus \{0\}$ for any LCA group Γ.

For the discrete Abelian group X, the algebraic Peters monoid is $\mathcal{M}_{\mathrm{alg}}(X) := (\mathcal{M}(X), *, \chi_{\{0\}})$. Define $w_{\mathrm{alg}} : \mathcal{M}_{\mathrm{alg}}(X) \to \mathbb{R}_{\geq 0}$, by

$$w_{\mathrm{alg}}(\phi) = \log(||\phi||_1/\phi(0)) \quad \text{for} \quad \phi \in \mathcal{M}(X).$$

This makes sense since $||\phi||_1 = \sum_{x \in X} \phi(x) \geq \phi(0) \neq 0$.

Lemma

In the above notation:

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Dually, for a compact Abelian group K, the topological Peters monoid is $\mathcal{M}_{\rm top}(K) = (\mathcal{M}(K), \cdot, \chi_K)$. Define the norm

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For a LCA group Γ the *Fourier transform* $\widehat{\phi}\colon \widehat{\Gamma} \to \mathbb{C}$ of $\phi \in L^1(\Gamma)$ is defined by

$$\widehat{\phi}(\gamma) = (\phi * \gamma)(0) = \int_{y\in\Gamma} \phi(y)\gamma(-y)\delta\mu(y) = \int_{y\in\Gamma} \phi(y)\overline{\gamma(y)}\delta\mu(y),$$

for $\gamma \in \Gamma^\wedge$.

Theorem

If X is a discrete abelian group and $K = X^\wedge$, then the Fourier transform

$$\widehat{(-)}\colon \mathcal{M}_{\mathrm{alg}}(X) \to \mathcal{M}_{\mathrm{top}}(K), \qquad \phi \mapsto \widehat{\phi}$$

is an isomorphism of normed monoids. Hence, $\widehat{\chi_{\{0\}}} = \chi_K$, $\widehat{\phi * \psi} = \widehat{\phi} \cdot \widehat{\psi}$ and $w_{\mathrm{alg}}(\phi) = w_{\mathrm{top}}(\widehat{\phi})$, for all $\phi, \psi \in \mathcal{M}_{\mathrm{alg}}(X)$.

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$$\widehat{\phi}(\gamma) = (\phi * \gamma)(0) = \int_{y\in\Gamma} \phi(y)\gamma(-y)\delta\mu(y) = \int_{y\in\Gamma} \phi(y)\overline{\gamma(y)}\delta\mu(y),$$

for $\gamma \in \Gamma^\wedge$.

Theorem

If X is a discrete abelian group and $K = X^\wedge$, then the Fourier transform

$$\widehat{(-)}\colon \mathcal{M}_{\mathrm{alg}}(X) \to \mathcal{M}_{\mathrm{top}}(K), \qquad \phi \mapsto \widehat{\phi}$$

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In the sequel G is an amenable group. For a right linear action $K \overset{\rho}{\curvearrowleft} G$ on a compact abelian group K the left action $G \overset{\rho_{\mathrm{top}}}{\curvearrowright} \mathcal{M}_{\mathrm{top}}(K)$, defined by $(\rho_{\mathrm{top}})_g(\phi) = \phi \circ \rho_g$ ($\phi \in \mathcal{M}_{\mathrm{top}}(X)$, $g \in G$), is an action by isomorphisms of normed monoids.

Similarly, for a discrete abelian group X and left linear action $G \overset{\lambda}{\curvearrowright} X$ the action

$$G \overset{\lambda_{\mathrm{alg}}}{\curvearrowright} \mathcal{M}_{\mathrm{alg}}(X), \qquad \text{such that} \qquad (\lambda_{\mathrm{alg}})_g(\phi) = \phi \circ \lambda_g^{-1},$$

for all $\phi \in \mathcal{M}_{\mathrm{alg}}(X)$ and $g \in G$, is well-defined.

Proposition (Justin Peters' equality)

For a left linear action $G \overset{\lambda}{\curvearrowright} X$ on a discrete abelian group X, $K = X^\wedge$ and the dual action $K \overset{\rho=\lambda^\wedge}{\curvearrowleft} G$ the G-actions $G \overset{\lambda_{\mathrm{alg}}}{\curvearrowright} \mathcal{M}_{\mathrm{alg}}(X)$ and $G \overset{\rho_{\mathrm{top}}}{\curvearrowright} \mathcal{M}_{\mathrm{top}}(K)$ are conjugated via the isomorphism of normed monoids induced by the Fourier transform $\widehat{(-)}: \mathcal{M}_{\mathrm{alg}}(X) \to \mathcal{M}_{\mathrm{top}}(K)$, $\phi \mapsto \widehat{\phi}$. Hence, $h(\lambda_{\mathrm{alg}}, \mathfrak{s}) = h(\rho_{\mathrm{top}}, \mathfrak{s})$ for every Følner net $\mathfrak{s}$ for G.

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The Bridge Theorem for amenable group actions

In the sequel X a discrete Abelian group with a left G-action $G \overset{\lambda}{\curvearrowright} X$, $K = X^\wedge$ and $K \overset{\rho}{\curvearrowleft} G$, with $\rho = \lambda^\wedge$.

Proposition

- (a) $G \overset{\lambda_{\mathfrak{F}}}{\curvearrowright} \mathfrak{F}(X)$ *and* $G \overset{\lambda_{\mathrm{alg}}}{\curvearrowright} \mathcal{M}_{\mathrm{alg}}(X)$ *are asymptotically equivalent.*
- (b) $G \overset{\rho_{\mathfrak{U}}}{\curvearrowright} \mathfrak{U}(K)$ *and* $G \overset{\rho_{\mathrm{top}}}{\curvearrowright} \mathcal{M}_{\mathrm{top}}(K)$ *are asymptotically equivalent.*

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Reduction to actions on compact spaces by surjective maps

For a right action $K \overset{\rho}{\curvearrowleft} S$ of a cancellative right amenable monoid S on a compact Hausdorff space K, we build (in 2 steps) its Ore colocalization $K^* \overset{\rho^*}{\curvearrowleft} G$, where K^* is a compact Hausdorff space and G is the group of left fractions of S. This construction preserves the topological entropy and linearity.

The surjective core of $K \overset{\rho}{\curvearrowleft} S$ is the closed S-invariant subspace $\bar{K} = E(\rho) := \bigcap_{t \in S} \rho_t(K) \overset{\varepsilon_K}{\hookrightarrow} K$ of K.

The restriction $\bar{\rho}_s := \rho_s \restriction_{\bar{K}} : \bar{K} \to \bar{K}$ is surjective for all $s \in S$.

Theorem (reduction to actions by surjective maps)

1. $h_{\mathrm{top}}(\bar{\rho}) = h_{\mathrm{top}}(\rho)$ for the restricted action $\bar{K} \overset{\bar{\rho}}{\curvearrowleft} S$.

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1. $h_{\mathrm{top}}(\bar{\rho}) = h_{\mathrm{top}}(\rho)$ for the restricted action $\bar{K} \overset{\bar{\rho}}{\curvearrowleft} S$.

2. this reduction is functorial, i.e., if $K' \overset{\rho'}{\curvearrowleft} S$ is an action on a compact Hausdorff space K' and $\phi \colon K \to K'$ is an S-equivariant continuous map, then $\phi(\bar{K}) \subseteq \bar{K}'$ and the continuous S-equivariant map $\bar{\phi} = \phi \restriction_{\bar{K}} : \bar{K} \to \bar{K}'$ is injective (resp., surjective), whenever ϕ is is injective (resp., surjective).

A by-product towards measure entropy

According to the well-known Halmos' paradigm, a continuous endomorphism $f : K \to K$ of a compact group is measure-preserving with respect to the Haar measure of K if and only if f is surjective.

Therefore, when applied to a right linear action $K \overset{\rho}{\curvearrowleft} S$ on a compact group K, the above theorem allows us to pass from ρ to the S-action $E(\rho) = \bar{K} \overset{\bar{\rho}}{\curvearrowleft} S$ by surjective continuous endomorphisms, hence measure-preserving maps.

In particular, one can also discuss the measure entropy of such an action; it is known that when $S = G$ is a countable amenable group and K is a compact metrizable group, then the topological and the measure entropy coincide.

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The Ore colocalization $K^* \overset{\rho^*}{\curvearrowleft} G$ of $K \overset{\rho}{\curvearrowleft} G$

For the inverse system $\mathfrak{K} = \{(K_g, \bar{\rho}_s \colon K_g \to K_{gs}) : g \in G,\ s \in S\}$, where $K_g = \bar{K}$ for all $g \in G$, let $K^* := \varprojlim \mathfrak{K}$. The canonical map $\pi_g = \pi_g^K \colon K^* \to K_g$ is surjective for all $g \in G$.
For $g \in G$ let $\rho_g^* \colon K^* \to K^*$ be the unique possible continuous map such that the following diagram commutes for all $h \in G$:

$$\begin{array}{ccc} K_{gh} = \bar{K} & \xrightarrow{\ \mathrm{id}_{\bar{K}}\ } & \bar{K} = K_h \\ \uparrow{\scriptstyle \pi_{gh}} & & \uparrow{\scriptstyle \pi_h} \\ K^* & \xrightarrow[\ \rho_g^*\ ]{} & K^*. \end{array}$$

This defines a right G-action $K^* \overset{\rho^*}{\curvearrowleft} G$, named (left) Ore colocalization of $K \overset{\rho}{\curvearrowleft} S$.

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Lemma

1. $\pi_1\colon K^* \to \bar{K}$ is (surjective and) S-equivariant, when K^* is endowed with the restriction $(\rho^*)_{\restriction S}$ of the action ρ^* to $S \leq G$;

2. the Ore colocalization is functorial, i.e., if $K' \overset{\rho'}{\curvearrowleft} S$ is an action on a compact Hausdorff space K' and $\phi\colon K \to K'$ is an S-equivariant continuous map, then there is a unique continuous map $\phi^*\colon K^* \to (K')^*$ such that, for every $g \in G$, the following diagram commutes

$$\begin{array}{ccc} K^* & \xrightarrow{\ \phi^*\ } & (K')^* \\ {\scriptstyle \varepsilon'_K \circ \pi_g^K}\downarrow & & \downarrow{\scriptstyle \varepsilon'_K \circ \pi_g^{K'}} \\ K & \xrightarrow[\ \phi\ ]{} & K'. \end{array}$$

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Theorem (Invariance under Ore colocalization)

Let $K \overset{\rho}{\curvearrowleft} S$ be a right S-action by continuous self-maps on a compact Hausdorff space K. Then $h_{\mathrm{top}}(\rho) = h_{\mathrm{top}}(\bar{\rho}) = h_{\mathrm{top}}(\rho^*)$.

Lemma (exactness of the Ore colocalization of linear actions)

Let $K \overset{\rho}{\curvearrowleft} S$ be a linear S-action on a compact group K, $H \leq K$ be a closed S-invariant subgroup and let $H \overset{\rho_H}{\curvearrowleft} S$ and $K/H \overset{\rho_{K/H}}{\curvearrowleft} S$ be the S-actions induced by ρ on H and on the left coset space K/H, respectively. If $\iota\colon H \to K$ is the inclusion and $\pi\colon K \to K/H$ the projection, then:

1. *the action $H^* \overset{(\rho_H)^*}{\curvearrowleft} G$ is conjugated to the action $\iota^*(H^*) \overset{(\rho^*)_{\iota^*(H^*)}}{\curvearrowleft} G$;*
2. *$\pi^*\colon K^* \to (K/H)^*$ is a surjective, G-equivariant, continuous and open map; moreover, the action $(K/H)^* \overset{(\rho_{K/H})^*}{\curvearrowleft} G$ is conjugated to the action $K^*/H^* \overset{(\rho^*)_{K^*/H^*}}{\curvearrowleft} G$ induced by ρ^* on the space of left H^*-cosets.*

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Reduction to the case of actions by injective maps

For a left linear action $S \overset{\lambda}{\curvearrowright} X$ on a discrete Abelian group X, we construct (again in 2 steps) its Ore localization $G \overset{\lambda^*}{\curvearrowright} X^*$, which is linear and preserves the algebraic entropy (i.e., $h_{\mathrm{alg}}(\lambda) = h_{\mathrm{alg}}(\lambda^*)$).

Starting with a left S-action $S \overset{\lambda}{\curvearrowright} X$ on an Abelian group X, define $\mathrm{Ker}(\lambda) := \{x \in X : \exists s \in S,\ \lambda_s(x) = 0\}$. This is a subgroup of X with $\lambda_s^{-1}(\mathrm{Ker}(\lambda)) = \mathrm{Ker}(\lambda)$, for all $s \in S$ (so, in particular, S-invariant). Let $\bar{X} := X/\mathrm{Ker}(\lambda)$ and $\pi_X : X \to \bar{X}$ be the canonical projection. Define a new left S-action $S \overset{\bar{\lambda}}{\curvearrowright} \bar{X}$ by letting $\bar{\lambda}_s(\pi(x)) = \pi(\lambda_s(x))$ for all $s \in S$ and $x \in X$. Then

1. $\bar{\lambda}$ acts on $\bar{X}$ by injective endomorphisms (i.e., $\bar{\lambda}_s$ is injective for all $s \in S$) and $h_{\mathrm{alg}}(\bar{\lambda}) = h_{\mathrm{alg}}(\lambda)$;
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2. this reduction is functorial, i.e., if $S \overset{\lambda'}{\curvearrowright} X'$ is an action on an Abelian group X' and $\phi \colon X \to X'$ is an S-equivariant homomorphism, then there is a unique homomorphism $\bar{\phi} \colon \bar{X} \to \bar{X}'$ with $\pi_{X'} \circ \phi = \bar{\phi} \circ \pi_X$; and $\bar{\phi}$ is injective (resp., surjective), whenever ϕ is injective (resp., surjective).

The Ore localization $G \overset{\lambda^*}{\curvearrowright} X^*$ of $S \overset{\lambda}{\curvearrowright} X$

Definition. With $S, X, \lambda, \bar{X}$ and $\bar{\lambda}$ as above consider the direct system of Abelian groups:

- $\mathfrak{X} := \{(X_g, \varepsilon_{gs,g} \colon X_{gs} \to X_g) : g \in G,\ s \in S\}$, where $X_g := \bar{X}$ and $\varepsilon_{gs,g} := \bar{\lambda}_s \colon \bar{X} \to \bar{X}$, for all $s \in S$ and $g \in G$;
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As in the case of colocalzation, there is a unique G-action $G \overset{\lambda^*}{\curvearrowright} X^*$, named Ore localization of $S \overset{\lambda}{\curvearrowright} X$, and $\varepsilon_1 \colon \bar{X} \to X^*$ is S-equivariant.

Lemma (The Ore localization is functorial)

The Ore localization $G \overset{\lambda^*}{\curvearrowright} X^*$ of $S \overset{\lambda}{\curvearrowright} X$ is functorial and the assignment $\phi \mapsto \phi^*$ preserves injectivity and surjectivity.

Theorem (Invariance under Ore localization)

In the above setting, $h_{\mathrm{alg}}(\lambda) = h_{\mathrm{alg}}(\bar{\lambda}) = h_{\mathrm{alg}}(\lambda^*)$.

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Proof of the Bridge Theorem

First we need the following:

Lemma

Given a left S-action $S \overset{\lambda}{\curvearrowright} X$ on a discrete Abelian group X, let $K := X^\wedge \overset{\rho := \lambda^\wedge}{\curvearrowleft} S$ be the right S-action induced by λ on the dual compact Abelian group $K := X^\wedge$.

1. $\mathrm{Ker}(\lambda)^\perp = E(\rho) \leq K$. *Furthermore, $\bar{\lambda}^\wedge$ is conjugated to $\bar{\rho}$.*
2. *Let $G \overset{\lambda^*}{\curvearrowright} X^*$ be the Ore localization of λ, $K := X^\wedge \overset{\rho := \lambda^\wedge}{\curvearrowleft} S$ the right S-action induced by λ on the dual, and $K^* \overset{\rho^*}{\curvearrowleft} G$ the Ore colocalization of ρ. Then, $K^* \overset{\rho^*}{\curvearrowleft} G$ is conjugated to $K^* \overset{(\lambda^*)^\wedge}{\curvearrowleft} S$*

Item (2), roughly speaking, says that, the Ore (co-)localization and the dual "commute" up to conjugacy, i.e., $(\lambda^{*\wedge}$ is conjugated to $(\lambda^\wedge)^*)$.

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By the invariance of entropy w.r.t. Ore (co-)localization

$$h_{\mathrm{top}}(\rho) = h_{\mathrm{top}}(\rho^*) \qquad \text{and} \qquad h_{\mathrm{alg}}(\lambda) = h_{\mathrm{alg}}(\lambda^*).$$

By item (2) of the previous lemma, $(\lambda^*)^{\wedge} = ((\rho^{\wedge})^*)^{\wedge}$ is conjugated to $(\rho^{\wedge\wedge})^*$, which is obviously conjugated to ρ^*. Hence, $h_{\mathrm{top}}(\rho^*) = h_{\mathrm{top}}((\lambda^*)^{\wedge})$. By the Bridge Theorem for actions of amenable groups,

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Applications: Proof of the Addition Theorems for $h_{\rm top}$ and $h_{\rm alg}$

Theorem (Addition Theorem for $h_{\rm top}$)

For a right linear action $K \overset{\rho}{\curvearrowleft} S$ on a compact group K and a ρ-invariant closed subgroup H of K the S-actions actions ρ_H and $\rho_{K/H}$ (induced by ρ on H and on the left cosets space K/H, respectively) satisfy

$$h_{\rm top}(\rho) = h_{\rm top}(\rho_H) + h_{\rm top}(\rho_{K/H}).$$

This was known for $\mathbb{N}$-actions as well as for actions of countable amenable groups on compact metrizable groups with H normal [Li].

Proof. First assume that $S = G$ is a group. Consider the diagonal action $(\rho_H)_{\rm cov} \oplus (\rho_{K/H})_{\rm cov}$ of G on ${\rm cov}(H) \oplus {\rm cov}(K/H)$, having as norm the sum of the respective norms. Since the norms of ${\rm cov}(H)$ and ${\rm cov}(K/H)$ (hence, of ${\rm cov}(H) \oplus {\rm cov}(K/H)$ as well) are sub-additive (so the $\mathfrak{s}$-entropy is a limit), one obtains

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Since the quantities in (*) and (**) do not depend on $\mathfrak{s}$, these equalities, along with the definition of $h_{\rm top}$ give

$$h_{\rm top}(\rho) = h(\rho_{\rm cov}, \mathfrak{s}) = h((\rho_H)_{\rm cov}, \mathfrak{s}) + h((\rho_{K/H})_{\rm cov}, \mathfrak{s}) =$$

$$h_{\rm top}(\rho_H) + h_{\rm top}(\rho_{K/H})$$

as required.

This ends the proof in the case case $S = G$ is a group.

(Continuation of Proof, general case.)

Let $G = S^{-1}S$ be the group of left fractions of S.
By the exactness of the Ore colocalization, we can identify H^* with a closed ρ-invariant subgroup of K^* (so that it makes sense to consider the space of left H^*-cosets K^*/H^*), and we can identify K^*/H^* with $(K/H)^*$. By the previous case

$$h_{\mathrm{top}}(\rho^*) = h_{\mathrm{top}}((\rho^*)_{H^*}) + h_{\mathrm{top}}((\rho^*)_{K^*/H^*}). \qquad (\dagger)$$

In view of the above identifications,

$$h_{\mathrm{top}}((\rho^*)_{H^*}) = h_{\mathrm{top}}((\rho_H)^*) \quad \text{and} \quad h_{\mathrm{top}}((\rho^*)_{K^*/H^*}) = h_{\mathrm{top}}((\rho_{K/H})^*).$$

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Addition Theorem for $h_{\rm alg}$

From the Addition Theorem for $h_{\rm top}$ and the Bridge Theorem, we deduce now an Addition Theorem for $h_{\rm alg}$ for left actions $S \overset{\lambda}{\curvearrowright} X$ of a cancellative amenable monoid S on a discrete Abelian group X:

Theorem (Addition Theorem for $h_{\rm alg}$)

For a linear action $S \overset{\lambda}{\curvearrowright} X$ on an abelian group X and a λ-invariant closed subgroup Y of X the left S-actions actions λ_Y and $\lambda_{X/Y}$ (induced by λ on Y and the quotient X/Y, respectively) satisfy

$$h_{\rm alg}(\lambda) = h_{\rm alg}(\lambda_Y) + h_{\rm alg}(\lambda_{X/Y}).$$

So far direct proofs of this fact are known only under the hypotheses that either X is torsion (Fornasiero, Giordano Bruno, DD [2020]) or S is countable and locally monotileable (Fornasiero, Giordano Bruno, Salizzoni, DD [2022]).

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Proof. Consider the compact Abelian group $K := X^\wedge$, its closed subgroup $H := Y^\perp$ and its quotient group $K/H \cong Y^\wedge$.
If $\rho := \lambda^\wedge$, then H is ρ-invariant and the action ρ_H induced by ρ on H by restriction is conjugated to $(\lambda_{X/Y})^\wedge$, while the right S-action $\rho_{K/H}$ induced by ρ on K/H is conjugated to $(\lambda_Y)^\wedge$.
Therefore, one can now conclude via the following series of equalities:

$$\begin{aligned}
h_{\mathrm{alg}}(\lambda) &= h_{\mathrm{top}}(\rho) && \text{by the Bridge Theorem;}\\
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