S spaces and Moore-Mrowka with large continuum

Alan Dow[1] and Saharon Shelah[2]

[1]footnotes not allowed

[2]except this one

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Question

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thanks for your attention

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It is independent if any of these exist but let's dig deeper

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[Dow, van Douwen] There are no Iw1-spaces.

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1. *[Stevo] there are no S spaces (more on this later)*
2. *[Balogh] there are no Moore-Mrowka spaces and therefore no lw1-spaces.*

what about Martin's Axiom and $\mathfrak{c} = \aleph_2$?

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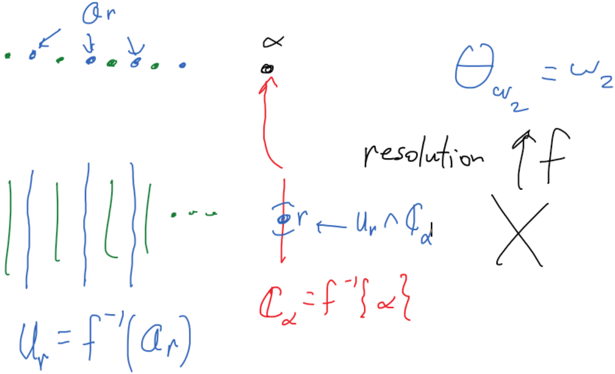
[KJS] adapt earlier Koszmider techniques to construct a finite condition (i.e. absolute) ccc poset Q_0 that not only adds Θ_{ω_2}

but also a resolution $f : X \mapsto \Theta_{\omega_2}$ so that X is first countable Iw1 with very special properties.

bad picture



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tightness of Θ_{ω_2}?? no need! the character of X is preserved by any poset. And that's how we get Martin's Axiom

Stevo's original 1982 proof of no S spaces

and a model with a Moore-Mrowka space and no S spaces!

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Given $\vec{W} = \{W_\alpha : \alpha < \omega_1\} \subset [\omega_1]^{<\aleph_1}$ define $Q_{\vec{W}} \subset [\omega_1]^{<\aleph_0}$ where $\alpha \neq \beta \in q \in Q_{\vec{W}}$ implies $\alpha \notin W_\beta$, and ordered by $\supset$

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Remark

$Q_{\vec{W}}$ is designed to force a discrete subset

For Moore-Mrowka just change to

$$\alpha < \beta \in q \quad \text{implies} \quad \alpha \in W_\beta$$

to force a free sequence

Hence my view that the problems are similar.

forcing tools

Remark

$Q_{\vec{W}}$ need not be ccc

recall $\mathrm{MA}(\omega_1)$ is consistent with there being S spaces

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Jensen's cub poset $\mathcal{J} = \{\langle a, A\rangle : a = \overline{a} \in [\omega_1]^{<\aleph_1}, A \subset \omega_1 \ cub\}$ and $(a, A) < (b, B)$ providing $b \subset a \subset b \cup B \setminus \max(b), A \subset B$

Let $C_{\mathcal{J}}$ denote the generic "fast" cub added

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$$Q_{\vec{W}}[C] = \{q \in Q_{\vec{W}} : \gamma \in C \to |q \cap (\gamma_C^+ \setminus \gamma)| \leq 1\}$$

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possibly even better: elementary submodels as side conditions

getting $Q_{\vec{W}}$ to be ccc

utilizing recent (in 1980) ideas of Avraham, Shelah, and Rubin

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Lemma (Stevo)

Let R be a ccc poset and let $\vec{W}$ be an S space sequence

$$\mathcal{C}_{\omega_1} * \dot{\mathcal{J}} \Vdash \check{R} * Q_{\vec{W}}[C_{\mathcal{J}}] \text{ is ccc}$$

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Lemma (Stevo)

Let $(\mathcal{C}_{\omega_1} * \dot{\mathcal{J}})_{\omega_2}$ be the mixed finite/countable support iteration. Let R be a ccc poset.

$(\mathcal{C}_{\omega_1} * \dot{\mathcal{J}})_{\omega_2}$ is proper and forces that R remains ccc.

here’s what happens next

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$\left(\mathcal{C}_{\omega_1} * \dot{\mathcal{J}}\right)_\lambda * \langle \dot{Q}_\beta : \beta < \lambda \rangle$ (tail is ccc – call it R)

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made some room

$\left(\mathcal{C}_{\omega_1} * \dot{\mathcal{J}}\right)_\lambda *$ ... $\langle \dot{Q}_\beta : \beta < \lambda \rangle$ (ccc)

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$$\left(\mathcal{C}_{\omega_1} * \dot{\mathcal{J}}\right)_\lambda * \left(\mathcal{C}_{\omega_1} * \dot{\mathcal{J}}\right)_{\omega_2 \setminus \lambda} * \langle \dot{Q}_\beta : \beta < \lambda \rangle$$ (still ccc)

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then jump back to $\left(\mathcal{C}_{\omega_1} * \dot{\mathcal{J}}\right)_{\lambda+1} * \langle \dot{Q}_\beta : \beta < \lambda \rangle$ to choose $\dot{Q}_\lambda$

to continue the recursive construction of $\mathbb{P}_{\omega_2+\omega_2}$

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[yep, still Stevo] Then with $C_{\mathcal{J},\lambda}$ being the cub at stage λ of $(\mathcal{C}_{\omega_1} * \dot{\mathcal{J}})_{\omega_2}$ and with $\dot{Q}_\lambda = Q_{\vec{W}}[C_{\mathcal{J},\lambda}]$

$$(\mathcal{C}_{\omega_1} * \dot{\mathcal{J}})_{\omega_2} * \langle \dot{Q}_\beta : \beta \leq \lambda \rangle \quad \text{is ccc}$$

for MA($\aleph_1$): often for $\alpha < \omega_2$ let $\dot{Q}_\alpha$ be the next small ccc poset .

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This gives no S spaces, MA, and Q_0 gives a Moore-Mrowka

now for $\mathfrak{c} = \kappa > \aleph_2$ with suitable $\diamondsuit$

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Assume, by induction, that for some $\lambda < \kappa$, for each $\alpha < \lambda$, $\dot{Q}_\alpha$ is a $(\mathcal{C}_{\omega_1} * \dot{\mathcal{J}})_\alpha * \langle \dot{Q}_\beta : \beta < \alpha \rangle$-name of a ccc poset

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Theorem

If $\vec{W}$ is a $(\mathcal{C}_{\omega_1} * \dot{\mathcal{J}})_\lambda * \langle \dot{Q}_\beta : \beta < \lambda \rangle$-name of an S space sequence, then there is an $\alpha \geq \lambda$ so that

$$(\mathcal{C}_{\omega_1} * \dot{\mathcal{J}})_\kappa \Vdash \langle \dot{Q}_\beta : \beta < \alpha \rangle * Q[C_{\mathcal{J},\alpha}] \text{ is ccc}$$

where $\dot{Q}_\beta$ $\lambda \leq \beta < \alpha$ can be, e.g. , $\mathcal{C}_\omega$

and therefore can ensure no S spaces.

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with still more effort we can also ensure there are no Moore-Mrowka spaces with cardinality greater than κ (i.e. $\mathfrak{c}$). Much harder since we are still trying to *kill* with $\aleph_1$-sized posets.