Planar absolute retracts and countable structures

Jan Dudák

Charles University in Prague

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- Borsuk: A locally connected continuum $X \subseteq \mathbb{R}^2$ is an AR if and only if $\mathbb{R}^2 \setminus X$ is connected.

Planar continua and their boundaries

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- In $\mathbb{R}^2$, this is not true anymore (consider the closed unit disc and the unit circle).
- Dudák, Vejnar: If $X_1, X_2 \subseteq \mathbb{R}^2$ are ARs such that ∂X_1 is homeomorphic to ∂X_2, then X_1 is homeomorphic to X_2.

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- Define a mapping $f\colon X_1 \to X_2$ by $f(x) = h(x)$ for $x \in \partial X_1$ and by $f(x) = g_C(x)$ for $x \in \mathrm{ins}(C)$, $C \in \mathcal{S}_1$.

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- It can be shown that f is a homeomorphism.

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- Fact: If X is a Polish space and B is a Borel subset of X, then B equipped with the σ-algebra $\{A \subseteq B\,;\, A \in \mathrm{Borel}(X)\}$ is a standard Borel space.

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- If both $E \leq_B F$ and $F \leq_B E$ hold true, we say that E is Borel bireducible with F.
- Many important equivalence relations (ERs) in mathematics can be naturally represented by ERs on suitable standard Borel spaces. The notion of Borel reducibility can then be applied.

Examples of known results

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- Ferenczi, Louveau, Rosendal: The isomorphism ER of separable Banach spaces is Borel bireducible with the universal analytic ER.

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- Becker, Kechris: Let S_∞ be the symmetric group on $\mathbb{N}$. An equivalence relation E on a standard Borel space X is classifiable by countable structures if and only if there is a standard Borel space Y and a Borel measurable action φ of S_∞ on Y such that E is Borel reducible to the orbit ER induced by φ.

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- Chang, Gao: Let $n \in \mathbb{N}$. Then the homeomorphism ER of compact sets in $\mathbb{R}^n$ is classifiable by countable structures if and only if $n = 1$.

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- Dudák, Vejnar: The homeomorphism ER of ARs in $\mathbb{R}^2$ is classifiable by countable structures.

Non-classification results and open problems

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- Question 1: Is the homeomorphism ER of absolute neighborhood retracts in $\mathbb{R}^2$ classifiable by countable structures?
- Question 2: Is it true that the homeomorphism ER of compact sets in $\mathbb{R}^2$ is strictly less complex than the homeomorphism ER of metrizable compact spaces?

Thank you for your attention.