Separable reductions, rich families, and projectional skeletons in non-separable Banach spaces

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Proof: Put $\mathcal{R} := \{Y \in \mathcal{S}(X) :\ \operatorname{diam} f(B(x,r) \cap Y) = \operatorname{diam} f(B(x,r))$ for every $x \in Y$ and every $r > 0\}$.

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Homework. Prove that if $\mathcal{R}_1, \mathcal{R}_2, \ldots$ are rich then $\mathcal{R}_1 \cap \mathcal{R}_2 \cap \ldots$ is rich.

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Warning. The separable reduction does not work for Gateaux differentiability —take a nowhere Gateaux differentiable norm on $\ell_\infty(\Gamma)$.

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Definition.

A projectional resolution of the identity (PRI) [J. Lindenstrauss 1965] on a Banach space $(X, \|\cdot\|)$ is a family $(P_\alpha : \omega \le \alpha \le \operatorname{dens} X)$ of linear bounded projections on X such that $P_\omega = 0$, $P_{\operatorname{dens} X}$ is the identity mapping, and for all $\omega \le \alpha \le \operatorname{dens} X$ the following hold:

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(iii) $\alpha \ne \omega \implies \overline{\bigcup_{\beta<\alpha} P_{\beta+1} X} = P_\alpha X$.

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Definition. A projectional skeleton (PS) [W. Kubiś 2009] in $(X, \|\cdot\|)$ is a family of linear bounded projections $\left(P_s :\ s \in \Gamma\right)$ on X, indexed by a partially ordered, up-directed, and σ-complete set $(\Gamma, \le)$, such that

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Definition. A projectional skeleton (PS) [W. Kubiś 2009] in $(X, \|\cdot\|)$ is a family of linear bounded projections $\big(P_s : \ s \in \Gamma\big)$ on X, indexed by a partially ordered, up-directed, and σ-complete set $(\Gamma, \le)$, such that

(i) $P_s X$ is `separable` for every $s \in \Gamma$,

(ii) $X = \bigcup_{s\in\Gamma} P_s X$,

(iii) $P_t \circ P_s = P_s = P_s \circ P_t$ whenever $s, t \in \Gamma$ and $s \le t$, and

(iv) If $s_1 < s_2 < \cdots$ in Γ, then $P_{\sup_{n\in\mathbb{N}} s_n} X = \overline{\bigcup_{n\in\mathbb{N}} P_{s_n} X}$.

Return back to Theorem 2

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What does Γ usually look like? Frequently they are rich families (see Theorem 2)

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e.g. $L_1(\mu)$, with any σ-additive measure μ, duals to C^* algebras, order continuous lattices, $C(G)$, with G a compact abelian group, and preduals of semifinite von Neumann algebras; see [O. Kalenda 2008].

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(iii)⇒(i) Needs a longer but not deep work (via transfinite induction).

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Similarly, we can characterize the classes of weakly $\mathcal{K}$-analytic spaces and of Vašák (i.e. WCD) spaces, see [C. Correa, M. Cúth, J. Somaglia 2022].

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Question (W. Kubiś). To characterize Banach spaces which are simultaneously Asplund and 1-Plička via suitable projectional skeletons.

Theorem 7 (C. Correa, M. Cúth, J. Somaglia 2022)

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