Canonical Ramsey Theory and Dynamics

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Background

van der Waerden's theorem([1]): For any $\ell \in \mathbb{N}$ and any finite partition $\mathbb{N} = \bigcup_{i=1}^{r} C_i$, there exists $a, d \in \mathbb{N}$ and $1 \le i_0 \le r$ for which $\{a+jd\}_{j=0}^{\ell} \subseteq C_{i_0}$.

Topological van der Waerden theorem([2],[3]): If (X,T) is a minimal topological dynamical system, $\emptyset \ne U \subseteq X$ is an open set, and $\ell \in \mathbb{N}$, then there exists $d \in \mathbb{N}$ such that

$$U \cap T^{-d}U \cap \cdots \cap T^{-\ell d}U \ne \emptyset \tag{1}$$

Szemerédi's theorem([4]): If $A \subseteq \mathbb{N}$ has positive upper density ($\limsup_{N\to\infty} \frac{1}{N}|A \cap [1,N]| > 0$), then A contains arbitrarily long arithmetic progressions.

Furstenberg multiple recurrence theorem([5],[3]): Let $(X, \mathscr{B}, \mu)$ be a probability space and $T : X \to X$ a measure preserving transformation. If $\ell \in \mathbb{N}$ and $A \in \mathscr{B}$ is such that $\mu(A) > 0$, then there exists $n \in \mathbb{N}$ for which

$$\mu(A \cap T^{-n}A \cap T^{-2n}A \cap \cdots \cap T^{-\ell n}A) > 0. \tag{2}$$

Canonical van der Waerden theorem([6],[7]): For any $\ell \in \mathbb{N}$ and any (not necessarily finite) partition $\mathbb{N} = \bigcup_{i=1}^{\infty} C_i$, $\exists\, a, d \in \mathbb{N}$ such that either
(i) $\{a+jd\}_{j=0}^{\ell} \subseteq C_{i_0}$ for some $i_0 \in \mathbb{N}$, or
(ii) for $0 \le j_1 < j_2 \le \ell$, $a + j_1 d$ and $a + j_2 d$ are not contained in the same cell (set of the form C_k).

Conjectures

Conjecture 1: Let X be a compact Hausdorff space and $T : X \to X$ a continuous map. For any open $U \subseteq X$ and any $\ell \in \mathbb{N}$, there exists $n \in \mathbb{N}$ with

$$U \cap T^{-n}U \cap T^{-2n}U \cap \cdots \cap T^{-\ell n}U \ne \emptyset, \text{ or} \tag{3}$$

$$T^{-in}U \cap T^{-jn}U = \emptyset\ \forall\ 0 \le i < j \le \ell. \tag{4}$$

Conjecture 2: Conjecture 1 holds if X is assumed to be a compact metric space.

Conjecture 3: For any $\ell \in \mathbb{N}$ and any (not necessarily finite) partition $\mathbb{N} = \bigcup_{i=1}^{\infty} C_i$, there exists $d \in \mathbb{N}$ such that either
(i) for some $i_0 \in \mathbb{N}$ we have $C_{i_0} \cap (C_{i_0} - d) \cap (C_{i_0} - 2d) \cap \cdots \cap (C_{i_0} - \ell d) \ne \emptyset$, or
(ii) for every $i \in \mathbb{N}$ we have $(C_i - jd) \cap (C_i - kd) = \emptyset$ for all $0 \le j < k \le \ell$.

Conjecture 4: Let $(X, \mathscr{B}, \mu)$ be a σ-finite measure space and $T : X \to X$ a measure preserving transformation. If $A \in \mathscr{B}$, then there exists $n \in \mathbb{N}$ with

$$\mu(A \cap T^{-n}A \cap T^{-2n}A \cap \cdots \cap T^{-\ell n}A) > 0, \text{ or} \tag{5}$$

$$\mu(T^{-in}A \cap T^{-jn}A) = 0\ \forall\ 0 \le i < j \le \ell. \tag{6}$$

Conjecture 5(Canonical Szemerédi)**:** For any $A \subseteq \mathbb{N}$ and $\ell \in \mathbb{N}$ there exists $d \in \mathbb{N}$ with
(i) $A \cap (A-d) \cap (A-2d) \cap \cdots \cap (A - \ell d) \ne \emptyset$, or
(ii) $(A - id) \cap (A - jd) = \emptyset$ for all $0 \le i < j \le \ell$.

Implications

-Conjecture 1 implies Conjectures 2, 3, 5, and Szemerédi's theorem. While van der Waerden's theorem is equivalent to its topological analogue, it is not clear whether or not Conjecture 3 implies Conjecture 1.
-Conjecture 2 implies the canonical van der Waerden theorem, but it is not obvious whether or not Conjecture 2 implies Conjecture 1, or whether the canonical van der Waerden theorem implies Conjecture 2.
-Conjecture 4 implies Conjecture 3, which implies Conjecture 5, which implies Szemerédi's theorem. While the Fursternberg multiple recurrence theorem is equivalent to Szemerédi's theorem, it is not clear whether or not Conjecture 4 implies Conjectures 2, or whether Conjecture 2 implies Conjecture 5.
-While the Fustenberg multiple recurrence theorem implies the topological van der Waerden Theorem, we do not know if Conjecture 1 implies Conjecture 4, or even if Conjecture 4 implies Conjecture 2.
Theorem: The following are equivalent to Conjecture 4:
(i) Conjecture 4 when $X = \mathbb{R}$, $\mathscr{B}$ is the Lebesgue σ-algebra, and μ is the Lebesgue measure.
(ii) Let $(X, \mathscr{B}, \mu)$ be a probability space and $T : X \to X$ a non-singular measurable transformation. If $A \in \mathscr{B}$, then there exists $n \in \mathbb{N}$ with

$$\mu(A \cap T^{-n}A \cap T^{-2n}A \cap \cdots \cap T^{-\ell n}A) > 0, \text{ or} \tag{7}$$

$$\mu(T^{-in}A \cap T^{-jn}A) = 0\ \forall\ 0 \le i < j \le \ell. \tag{8}$$

(iii) Item (ii) when $X = [0,1]$, $\mathscr{B}$ is the Lebesgue σ-algebra, and μ is the Lebesgue measure.

References

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