Universal Flows Revisited

Stefan Geschke
University of Hamburg

July 25, 2022

Introduction

Let G be a topological group.

A *G-flow* is a compact space X, the *phase space*, together with a continuous *action*

$$G \times X \to X; (g,x) \mapsto gx$$

satisfying $1_G x = x$ and

$$(g_1 g_2)x = g_1(g_2 x)$$

for all $g_1, g_2 \in T$ and all $x \in X$.

A map $f : X \to Y$ between two G-flows is *equivariant* if for all $x \in X$ and all $g \in G$ we have

$$f(gx) = gf(x).$$

A G-flow Y is a *factor* of a G-flow X if there is continuous equivariant map from X onto Y.

Introduction

Let G be a topological group.

A *G-flow* is a compact space X, the *phase space*, together with a continuous *action*

$$G \times X \to X; (g, x) \mapsto gx$$

satisfying $1_G x = x$ and

$$(g_1 g_2)x = g_1(g_2 x)$$

for all $g_1, g_2 \in T$ and all $x \in X$.

A map $f : X \to Y$ between two G-flows is *equivariant* if for all $x \in X$ and all $g \in G$ we have

$$f(gx) = gf(x).$$

A G-flow Y is a *factor* of a G-flow X if there is continuous equivariant map from X onto Y.

Introduction

Let G be a topological group.
A *G-flow* is a compact space X, the *phase space*, together with a continuous *action*

$$G \times X \to X; (g, x) \mapsto gx$$

satisfying $1_G x = x$ and

$$(g_1 g_2)x = g_1(g_2 x)$$

for all $g_1, g_2 \in T$ and all $x \in X$.
A map $f : X \to Y$ between two G-flows is *equivariant* if for all $x \in X$ and all $g \in G$ we have

$$f(gx) = gf(x).$$

A G-flow Y is a *factor* of a G-flow X if there is continuous equivariant map from X onto Y.

Introduction

Let G be a topological group.
A *G-flow* is a compact space X, the *phase space*, together with a continuous *action*

$$G \times X \to X; (g, x) \mapsto gx$$

satisfying $1_G x = x$ and

$$(g_1 g_2)x = g_1(g_2 x)$$

for all $g_1, g_2 \in T$ and all $x \in X$.
A map $f : X \to Y$ between two G-flows is *equivariant* if for all $x \in X$ and all $g \in G$ we have

$$f(gx) = gf(x).$$

A G-flow Y is a *factor* of a G-flow X if there is continuous equivariant map from X onto Y.

Introduction

Let G be a topological group.
A *G-flow* is a compact space X, the *phase space*, together with a continuous *action*

$$G \times X \to X; (g, x) \mapsto gx$$

satisfying $1_G x = x$ and

$$(g_1 g_2)x = g_1(g_2 x)$$

for all $g_1, g_2 \in T$ and all $x \in X$.
A map $f : X \to Y$ between two G-flows is *equivariant* if for all $x \in X$ and all $g \in G$ we have

$$f(gx) = gf(x).$$

A G-flow Y is a *factor* of a G-flow X if there is continuous equivariant map from X onto Y.

If $\mathcal{C}$ is a class of G-flows, then a flow $X \in \mathcal{C}$ is *universal* (for $\mathcal{C}$) if every $Y \in \mathcal{C}$ is a factor of X.

A G-flow X is *minimal* if X has no proper (closed) subflow or, equivalently, if every G-orbit $Gx = \{gx : g \in G\}$ of an element x of X is dense in X.

A straight forward application of Zorn's Lemma shows that every G-flow has a minimal subflow.

Now suppose that G is discrete. Then G acts continuously on its Čech-Stone compactification βG.

Any minimal subflow of βG is a universal minimal G-flow and any two universal minimal G-flows are homeomorphic by an equivariant map.

However, a minimal subflow of βG is typically of uncountable weight and therefore not metrizable.

If $\mathcal{C}$ is a class of G-flows, then a flow $X \in \mathcal{C}$ is *universal* (for $\mathcal{C}$) if every $Y \in \mathcal{C}$ is a factor of X.

A G-flow X is *minimal* if X has no proper (closed) subflow or, equivalently, if every G-orbit $Gx = \{gx : g \in G\}$ of an element x of X is dense in X.
A straight forward application of Zorn's Lemma shows that every G-flow has a minimal subflow.

Now suppose that G is discrete. Then G acts continuously on its Čech-Stone compactification βG.
Any minimal subflow of βG is a universal minimal G-flow and any two universal minimal G-flows are homeomorphic by an equivariant map.
However, a minimal subflow of βG is typically of uncountable weight and therefore not metrizable.

If $\mathcal{C}$ is a class of G-flows, then a flow $X \in \mathcal{C}$ is *universal* (for $\mathcal{C}$) if every $Y \in \mathcal{C}$ is a factor of X.

A G-flow X is *minimal* if X has no proper (closed) subflow or, equivalently, if every G-orbit $Gx = \{gx : g \in G\}$ of an element x of X is dense in X.
A straight forward application of Zorn's Lemma shows that every G-flow has a minimal subflow.

Now suppose that G is discrete. Then G acts continuously on its Čech-Stone compactification βG.
Any minimal subflow of βG is a universal minimal G-flow and any two universal minimal G-flows are homeomorphic by an equivariant map.
However, a minimal subflow of βG is typically of uncountable weight and therefore not metrizable.

If $\mathcal{C}$ is a class of G-flows, then a flow $X \in \mathcal{C}$ is *universal* (for $\mathcal{C}$) if every $Y \in \mathcal{C}$ is a factor of X.

A G-flow X is *minimal* if X has no proper (closed) subflow or, equivalently, if every G-orbit $Gx = \{gx : g \in G\}$ of an element x of X is dense in X.
A straight forward application of Zorn's Lemma shows that every G-flow has a minimal subflow.

Now suppose that G is discrete. Then G acts continuously on its Čech-Stone compactification βG.
Any minimal subflow of βG is a universal minimal G-flow and any two universal minimal G-flows are homeomorphic by an equivariant map.
However, a minimal subflow of βG is typically of uncountable weight and therefore not metrizable.

If $\mathcal{C}$ is a class of G-flows, then a flow $X \in \mathcal{C}$ is *universal* (for $\mathcal{C}$) if every $Y \in \mathcal{C}$ is a factor of X.

A G-flow X is *minimal* if X has no proper (closed) subflow or, equivalently, if every G-orbit $Gx = \{gx : g \in G\}$ of an element x of X is dense in X.
A straight forward application of Zorn's Lemma shows that every G-flow has a minimal subflow.

Now suppose that G is discrete. Then G acts continuously on its Čech-Stone compactification βG.
Any minimal subflow of βG is a universal minimal G-flow and any two universal minimal G-flows are homeomorphic by an equivariant map.
However, a minimal subflow of βG is typically of uncountable weight and therefore not metrizable.

If $\mathcal{C}$ is a class of G-flows, then a flow $X \in \mathcal{C}$ is *universal* (for $\mathcal{C}$) if every $Y \in \mathcal{C}$ is a factor of X.

A G-flow X is *minimal* if X has no proper (closed) subflow or, equivalently, if every G-orbit $Gx = \{gx : g \in G\}$ of an element x of X is dense in X.
A straight forward application of Zorn's Lemma shows that every G-flow has a minimal subflow.

Now suppose that G is discrete. Then G acts continuously on its Čech-Stone compactification βG.
Any minimal subflow of βG is a universal minimal G-flow and any two universal minimal G-flows are homeomorphic by an equivariant map.
However, a minimal subflow of βG is typically of uncountable weight and therefore not metrizable.

This brings up the question whether there are universal objects in the class of metric minimal flows.

There are infinite topological groups for which the one-point flow is the universal minimal flow. These are the *extremely amenable* groups.

If $G = \mathbb{Z}$, then there is no universal minimal metric flow. This follows from results of Furstenberg, Foreman, and Beleznay:

This brings up the question whether there are universal objects in the class of metric minimal flows.

There are infinite topological groups for which the one-point flow is the universal minimal flow. These are the *extremely amenable* groups.

If $G = \mathbb{Z}$, then there is no universal minimal metric flow. This follows from results of Furstenberg, Foreman, and Beleznay:

This brings up the question whether there are universal objects in the class of metric minimal flows.

There are infinite topological groups for which the one-point flow is the universal minimal flow. These are the *extremely amenable* groups.

If $G = \mathbb{Z}$, then there is no universal minimal metric flow. This follows from results of Furstenberg, Foreman, and Beleznay:

Furstenberg's structure theorem allows it to assign an ordinal rank to every minimal *distal* metric flow, the *distal height* of the flow.
This rank does not increase when taking factors. The the distal height of a minimal distal metric flow is countable.

Theorem (Foreman and Beleznay)

Every countable ordinal is the distal height of a minimal distal metric $\mathbb{Z}$-flow.

It follows that there is no universal minimal distal metric $\mathbb{Z}$-flow.
But every minimal metric flow has a maximal distal factor, which is also minimal.
Hence there is no universal minimal metric $\mathbb{Z}$-flow.

Furstenberg's structure theorem allows it to assign an ordinal rank to every minimal *distal* metric flow, the *distal height* of the flow. This rank does not increase when taking factors. The the distal height of a minimal distal metric flow is countable.

Theorem (Foreman and Beleznay)

Every countable ordinal is the distal height of a minimal distal metric $\mathbb{Z}$-flow.

It follows that there is no universal minimal distal metric $\mathbb{Z}$-flow. But every minimal metric flow has a maximal distal factor, which is also minimal.

Hence there is no universal minimal metric $\mathbb{Z}$-flow.

Furstenberg's structure theorem allows it to assign an ordinal rank to every minimal *distal* metric flow, the *distal height* of the flow. This rank does not increase when taking factors. The the distal height of a minimal distal metric flow is countable.

Theorem (Foreman and Beleznay)

Every countable ordinal is the distal height of a minimal distal metric $\mathbb{Z}$-flow.

It follows that there is no universal minimal distal metric $\mathbb{Z}$-flow. But every minimal metric flow has a maximal distal factor, which is also minimal.

Hence there is no universal minimal metric $\mathbb{Z}$-flow.

Furstenberg's structure theorem allows it to assign an ordinal rank to every minimal *distal* metric flow, the *distal height* of the flow. This rank does not increase when taking factors. The the distal height of a minimal distal metric flow is countable.

Theorem (Foreman and Beleznay)

Every countable ordinal is the distal height of a minimal distal metric $\mathbb{Z}$-flow.

It follows that there is no universal minimal distal metric $\mathbb{Z}$-flow. But every minimal metric flow has a maximal distal factor, which is also minimal.

Hence there is no universal minimal metric $\mathbb{Z}$-flow.

Furstenberg's structure theorem allows it to assign an ordinal rank to every minimal *distal* metric flow, the *distal height* of the flow. This rank does not increase when taking factors. The the distal height of a minimal distal metric flow is countable.

Theorem (Foreman and Beleznay)

Every countable ordinal is the distal height of a minimal distal metric $\mathbb{Z}$-flow.

It follows that there is no universal minimal distal metric $\mathbb{Z}$-flow.
But every minimal metric flow has a maximal distal factor, which is also minimal.
Hence there is no universal minimal metric $\mathbb{Z}$-flow.

Furstenberg's structure theorem allows it to assign an ordinal rank to every minimal *distal* metric flow, the *distal height* of the flow. This rank does not increase when taking factors. The the distal height of a minimal distal metric flow is countable.

Theorem (Foreman and Beleznay)

Every countable ordinal is the distal height of a minimal distal metric $\mathbb{Z}$-flow.

It follows that there is no universal minimal distal metric $\mathbb{Z}$-flow. But every minimal metric flow has a maximal distal factor, which is also minimal.

Hence there is no universal minimal metric $\mathbb{Z}$-flow.

Furstenberg's structure theorem allows it to assign an ordinal rank to every minimal *distal* metric flow, the *distal height* of the flow. This rank does not increase when taking factors. The the distal height of a minimal distal metric flow is countable.

Theorem (Foreman and Beleznay)

Every countable ordinal is the distal height of a minimal distal metric $\mathbb{Z}$-flow.

It follows that there is no universal minimal distal metric $\mathbb{Z}$-flow.
But every minimal metric flow has a maximal distal factor, which is also minimal.
Hence there is no universal minimal metric $\mathbb{Z}$-flow.

Zero-dimensional flows and Stone duality

The following generalizes a theorem of Anderson.

Theorem

Let G be a discrete group and let X be a G-flow of weight κ. Then X is a factor of a zero-dimensional G-flow of weight at most $|G| + \kappa$.

If X is a 0-dimensional G-flow, then we can consider the Boolean algebra $\mathrm{Clop}(X)$ of clopen subsets of X together with group action given by $ga = g^{-1}[a]$ for all $g \in G$ and $a \in \mathrm{Clop}(X)$.

Zero-dimensional flows and Stone duality

The following generalizes a theorem of Anderson.

Theorem

Let G be a discrete group and let X be a G-flow of weight κ. Then X is a factor of a zero-dimensional G-flow of weight at most $|G| + \kappa$.

If X is a 0-dimensional G-flow, then we can consider the Boolean algebra $\mathrm{Clop}(X)$ of clopen subsets of X together with group action given by $ga = g^{-1}[a]$ for all $g \in G$ and $a \in \mathrm{Clop}(X)$.

Zero-dimensional flows and Stone duality

The following generalizes a theorem of Anderson.

Theorem

Let G be a discrete group and let X be a G-flow of weight κ. Then X is a factor of a zero-dimensional G-flow of weight at most $|G| + \kappa$.

If X is a 0-dimensional G-flow, then we can consider the Boolean algebra $\mathrm{Clop}(X)$ of clopen subsets of X together with group action given by $ga = g^{-1}[a]$ for all $g \in G$ and $a \in \mathrm{Clop}(X)$.

Zero-dimensional flows and Stone duality

The following generalizes a theorem of Anderson.

Theorem

Let G be a discrete group and let X be a G-flow of weight κ. Then X is a factor of a zero-dimensional G-flow of weight at most $|G| + \kappa$.

If X is a 0-dimensional G-flow, then we can consider the Boolean algebra Clop(X) of clopen subsets of X together with group action given by $ga = g^{-1}[a]$ for all $g \in G$ and $a \in \text{Clop}(X)$.

If X and Y are G-flows and $h : X \to Y$ is continuous and equivariant, then $h^* : \mathrm{Clop}(Y) \to \mathrm{Clop}(X)$ defined by $h^*(a) = h^{-1}[a]$ for each $a \in \mathrm{Clop}(Y)$ is an equivariant homomorphism.

h is onto iff h^* is 1-1. Hence, by our theorem on lifting G-flows to zero-dimensional G-flows, if we are interested in universal objects, instead of looking at factors of metric flows, we can study embeddings of (countable) Boolean algebras with G-actions (G-Bas).

If X and Y are G-flows and $h : X \to Y$ is continuous and equivariant, then $h^* : \mathrm{Clop}(Y) \to \mathrm{Clop}(X)$ defined by $h^*(a) = h^{-1}[a]$ for each $a \in \mathrm{Clop}(Y)$ is an equivariant homomorphism.

h is onto iff h^* is 1-1. Hence, by our theorem on lifting G-flows to zero-dimensional G-flows, if we are interested in universal objects, instead of looking at factors of metric flows, we can study embeddings of (countable) Boolean algebras with G-actions (G-Bas).

Definition

If A is a G-Ba and $a \in A$, then by $\langle a \rangle_G$ we denote the smallest subalgebra B of A such that $a \in B$ and B is closed under the action by G. The Boolean algebra $\langle a \rangle_G$ is the subalgebra of A generated by the G-orbit of a.

Definition

Given two G-Bas A and B and elements $a \in A$ and $b \in B$, we call the pairs (A, a) and (B, b) *isomorphic* if there is an isomorphism between the G-Bas A and B that maps a to b.

Definition

Given a G-Ba A and $a \in A$, the *type* of a is the isomorphism type of the pair $(\langle a \rangle_G, a)$.

Definition

If A is a G-Ba and $a \in A$, then by $\langle a \rangle_G$ we denote the smallest subalgebra B of A such that $a \in B$ and B is closed under the action by G. The Boolean algebra $\langle a \rangle_G$ is the subalgebra of A generated by the G-orbit of a.

Definition

Given two G-Bas A and B and elements $a \in A$ and $b \in B$, we call the pairs (A, a) and (B, b) *isomorphic* if there is an isomorphism between the G-Bas A and B that maps a to b.

Definition

Given a G-Ba A and $a \in A$, the *type* of a is the isomorphism type of the pair $(\langle a \rangle_G, a)$.

Definition

If A is a G-Ba and $a \in A$, then by $\langle a \rangle_G$ we denote the smallest subalgebra B of A such that $a \in B$ and B is closed under the action by G. The Boolean algebra $\langle a \rangle_G$ is the subalgebra of A generated by the G-orbit of a.

Definition

Given two G-Bas A and B and elements $a \in A$ and $b \in B$, we call the pairs (A, a) and (B, b) *isomorphic* if there is an isomorphism between the G-Bas A and B that maps a to b.

Definition

Given a G-Ba A and $a \in A$, the *type* of a is the isomorphism type of the pair $(\langle a \rangle_G, a)$.

If A is a G-Ba and $I \subseteq A$ is an ideal that is closed under the action, then G acts on the quotient A/I.

On the other hand, the kernel of an equivariant homomorphism from a G-Ba A to a G-Ba B is an ideal that is closed under the action.

Definition

Let $\mathrm{Fr}(G)$ be the free Boolean algebra over the set $\{a_g : g \in G\}$ of generators. We assume that the a_g are pairwise distinct. G acts on the set of generators by letting $ha_g = a_{hg}$ This induces a G-action on $\mathrm{Fr}(G)$.

If A is a G-Ba and $I \subseteq A$ is an ideal that is closed under the action, then G acts on the quotient A/I.

On the other hand, the kernel of an equivariant homomorphism from a G-Ba A to a G-Ba B is an ideal that is closed under the action.

Definition

Let $\mathrm{Fr}(G)$ be the free Boolean algebra over the set $\{a_g : g \in G\}$ of generators. We assume that the a_g are pairwise distinct. G acts on the set of generators by letting $ha_g = a_{hg}$ This induces a G-action on $\mathrm{Fr}(G)$.

If A is a G-Ba and $I \subseteq A$ is an ideal that is closed under the action, then G acts on the quotient A/I.

On the other hand, the kernel of an equivariant homomorphism from a G-Ba A to a G-Ba B is an ideal that is closed under the action.

Definition

Let $\mathrm{Fr}(G)$ be the free Boolean algebra over the set $\{a_g : g \in G\}$ of generators. We assume that the a_g are pairwise distinct. G acts on the set of generators by letting $ha_g = a_{hg}$ This induces a G-action on $\mathrm{Fr}(G)$.

Lemma

Let A be a G-Ba and $a \in A$. Then there is a unique equivariant Boolean homomorphism $\pi : \mathrm{Fr}(G) \to A$ such that $\pi(a_{1_G}) = a$.

Lemma

Let A and B be G-Bas, $a \in A$, and $b \in B$. Suppose that $A = \langle a \rangle_G$ and $B = \langle b \rangle_G$.
Let $\pi_A : \mathrm{Fr}(G) \to A$ and $\pi_B : \mathrm{Fr}(G) \to B$ be the unique equivariant homomorphims with $\pi_A(a_{1_G}) = a$ and $\pi_B(a_{1_G}) = b$.
Then a and b have the same type iff the ideals $\pi_A^{-1}(0)$ and $\pi_B^{-1}(0)$ are identical.

Lemma

Let A be a G-Ba and $a \in A$. Then there is a unique equivariant Boolean homomorphism $\pi : \mathrm{Fr}(G) \to A$ such that $\pi(a_{1_G}) = a$.

Lemma

Let A and B be G-Bas, $a \in A$, and $b \in B$. Suppose that $A = \langle a \rangle_G$ and $B = \langle b \rangle_G$.
Let $\pi_A : \mathrm{Fr}(G) \to A$ and $\pi_B : \mathrm{Fr}(G) \to B$ be the unique equivariant homomorphims with $\pi_A(a_{1_G}) = a$ and $\pi_B(a_{1_G}) = b$. Then a and b have the same type iff the ideals $\pi_A^{-1}(0)$ and $\pi_B^{-1}(0)$ are identical.

Lemma

Let A be a G-Ba and $a \in A$. Then there is a unique equivariant Boolean homomorphism $\pi : \mathrm{Fr}(G) \to A$ such that $\pi(a_{1_G}) = a$.

Lemma

Let A and B be G-Bas, $a \in A$, and $b \in B$. Suppose that $A = \langle a \rangle_G$ and $B = \langle b \rangle_G$.
Let $\pi_A : \mathrm{Fr}(G) \to A$ and $\pi_B : \mathrm{Fr}(G) \to B$ be the unique equivariant homomorphims with $\pi_A(a_{1_G}) = a$ and $\pi_B(a_{1_G}) = b$.
Then a and b have the same type iff the ideals $\pi_A^{-1}(0)$ and $\pi_B^{-1}(0)$ are identical.

Lemma

Let A be a G-Ba and $a \in A$. Then there is a unique equivariant Boolean homomorphism $\pi : \mathrm{Fr}(G) \to A$ such that $\pi(a_{1_G}) = a$.

Lemma

Let A and B be G-Bas, $a \in A$, and $b \in B$. Suppose that $A = \langle a \rangle_G$ and $B = \langle b \rangle_G$.
Let $\pi_A : \mathrm{Fr}(G) \to A$ and $\pi_B : \mathrm{Fr}(G) \to B$ be the unique equivariant homomorphims with $\pi_A(a_{1_G}) = a$ and $\pi_B(a_{1_G}) = b$.
Then a and b have the same type iff the ideals $\pi_A^{-1}(0)$ and $\pi_B^{-1}(0)$ are identical.

Minimal subshifts of 2^G

Definition

On the space $\{0,1\}^G = 2^G$ we consider the G-action (*shift action*) defined as follows:
For all $g, h \in G$ and $x \in 2^G$ let $(hx)(g) = x(h^{-1}g)$.

Note that $\mathrm{Clop}(2^G)$ is isomorphic to $\mathrm{Fr}(G)$ by an equivariant isomorphism.

Minimal subshifts of 2^G

Definition

On the space $\{0,1\}^G = 2^G$ we consider the G-action (*shift action*) defined as follows:
For all $g, h \in G$ and $x \in 2^G$ let $(hx)(g) = x(h^{-1}g)$.

Note that $\mathrm{Clop}(2^G)$ is isomorphic to $\mathrm{Fr}(G)$ by an equivariant isomorphism.

Minimal subshifts of 2^G

Definition

On the space $\{0,1\}^G = 2^G$ we consider the G-action (*shift action*) defined as follows:
For all $g, h \in G$ and $x \in 2^G$ let $(hx)(g) = x(h^{-1}g)$.

Note that $\mathrm{Clop}(2^G)$ is isomorphic to $\mathrm{Fr}(G)$ by an equivariant isomorphism.

Minimal subshifts of 2^G

Definition

On the space $\{0,1\}^G = 2^G$ we consider the G-action (*shift action*) defined as follows:
For all $g, h \in G$ and $x \in 2^G$ let $(hx)(g) = x(h^{-1}g)$.

Note that $\mathrm{Clop}(2^G)$ is isomorphic to $\mathrm{Fr}(G)$ by an equivariant isomorphism.

The goal is now to construct many types of 1-generated G-Bas by looking at *subshifts* of 2^G.

Definition

We briefly consider the case $G = \mathbb{Z}$. A *Sturmian word* is a word $x \in \{0,1\}^{\mathbb{Z}}$ such that there are two real numbers, the *slope* α and the *intercept* ρ, with $\alpha \in [0,1)$ irrational such that for all $i \in \mathbb{Z}$ we have

$$x(i) = 1 \quad \Leftrightarrow \quad ((\rho + i \cdot \alpha) \mod 1) \in [0, \alpha).$$

It is well known that the orbit closure $C_x = \mathrm{cl}\{nx : n \in \mathbb{Z}\}$ of a Sturmian word with the restriction of the shift is a minimal $\mathbb{Z}$-flow. Also, the slope α can be computed from every element of the orbit closure of x.

The goal is now to construct many types of 1-generated G-Bas by looking at *subshifts* of 2^G.

Definition

We briefly consider the case $G = \mathbb{Z}$. A *Sturmian word* is a word $x \in \{0,1\}^{\mathbb{Z}}$ such that there are two real numbers, the *slope* α and the *intercept* ρ, with $\alpha \in [0,1)$ irrational such that for all $i \in \mathbb{Z}$ we have

$$x(i) = 1 \quad \Leftrightarrow \quad ((\rho + i \cdot \alpha) \mod 1) \in [0, \alpha).$$

It is well known that the orbit closure $C_x = \mathrm{cl}\{nx : n \in \mathbb{Z}\}$ of a Sturmian word with the restriction of the shift is a minimal $\mathbb{Z}$-flow. Also, the slope α can be computed from every element of the orbit closure of x.

The goal is now to construct many types of 1-generated G-Bas by looking at *subshifts* of 2^G.

Definition

We briefly consider the case $G = \mathbb{Z}$. A *Sturmian word* is a word $x \in \{0,1\}^{\mathbb{Z}}$ such that there are two real numbers, the *slope* α and the *intercept* ρ, with $\alpha \in [0,1)$ irrational such that for all $i \in \mathbb{Z}$ we have

$$x(i) = 1 \quad \Leftrightarrow \quad ((\rho + i \cdot \alpha) \mod 1) \in [0, \alpha).$$

It is well known that the orbit closure $C_x = \mathrm{cl}\{nx : n \in \mathbb{Z}\}$ of a Sturmian word with the restriction of the shift is a minimal $\mathbb{Z}$-flow. Also, the slope α can be computed from every element of the orbit closure of x.

The goal is now to construct many types of 1-generated G-Bas by looking at *subshifts* of 2^G.

Definition

We briefly consider the case $G = \mathbb{Z}$. A *Sturmian word* is a word $x \in \{0,1\}^{\mathbb{Z}}$ such that there are two real numbers, the *slope* α and the *intercept* ρ, with $\alpha \in [0,1)$ irrational such that for all $i \in \mathbb{Z}$ we have

$$x(i) = 1 \quad \Leftrightarrow \quad ((\rho + i \cdot \alpha) \mod 1) \in [0, \alpha).$$

It is well known that the orbit closure $C_x = \mathrm{cl}\{nx : n \in \mathbb{Z}\}$ of a Sturmian word with the restriction of the shift is a minimal $\mathbb{Z}$-flow. Also, the slope α can be computed from every element of the orbit closure of x.

It follows that for different irrational numbers $\alpha, \beta \in [0,1)$, Sturmian words of slope α and β have different (even disjoint) orbit closures.

We call the orbit closure of a Sturmian word together with the induced $\mathbb{Z}$-action a *Sturmian subshift*.

Given a Sturmian subshift X, we denote the common slope of all Sturmian words that generate X by $\alpha(X)$.

Lemma

Let X be a Sturmian subshift and let $\pi_X : \mathrm{Fr}(\mathbb{Z}) \to \mathrm{Clop}(X)$ be the homomorphism dual to the embedding of X into $2^{\mathbb{Z}}$. Then $\langle \pi_X(a_0) \rangle_{\mathbb{Z}} = \mathrm{Clop}(X)$ and the type of $\pi_X(a_0)$ determines $\alpha(X)$.

It follows that for different irrational numbers $\alpha, \beta \in [0, 1)$, Sturmian words of slope α and β have different (even disjoint) orbit closures.

We call the orbit closure of a Sturmian word together with the induced $\mathbb{Z}$-action a *Sturmian subshift*.

Given a Sturmian subshift X, we denote the common slope of all Sturmian words that generate X by $\alpha(X)$.

Lemma

Let X be a Sturmian subshift and let $\pi_X : \mathrm{Fr}(\mathbb{Z}) \to \mathrm{Clop}(X)$ be the homomorphism dual to the embedding of X into $2^{\mathbb{Z}}$. Then $\langle \pi_X(a_0) \rangle_{\mathbb{Z}} = \mathrm{Clop}(X)$ and the type of $\pi_X(a_0)$ determines $\alpha(X)$.

It follows that for different irrational numbers $\alpha, \beta \in [0, 1)$, Sturmian words of slope α and β have different (even disjoint) orbit closures.
We call the orbit closure of a Sturmian word together with the induced $\mathbb{Z}$-action a *Sturmian subshift*.

Given a Sturmian subshift X, we denote the common slope of all Sturmian words that generate X by $\alpha(X)$.

Lemma

Let X be a Sturmian subshift and let $\pi_X : \mathrm{Fr}(\mathbb{Z}) \to \mathrm{Clop}(X)$ be the homomorphism dual to the embedding of X into $2^{\mathbb{Z}}$. Then $\langle \pi_X(a_0) \rangle_{\mathbb{Z}} = \mathrm{Clop}(X)$ and the type of $\pi_X(a_0)$ determines $\alpha(X)$.

It follows that for different irrational numbers $\alpha, \beta \in [0, 1)$, Sturmian words of slope α and β have different (even disjoint) orbit closures.
We call the orbit closure of a Sturmian word together with the induced $\mathbb{Z}$-action a *Sturmian subshift*.

Given a Sturmian subshift X, we denote the common slope of all Sturmian words that generate X by $\alpha(X)$.

Lemma

Let X be a Sturmian subshift and let $\pi_X : \mathrm{Fr}(\mathbb{Z}) \to \mathrm{Clop}(X)$ be the homomorphism dual to the embedding of X into $2^{\mathbb{Z}}$. Then $\langle \pi_X(a_0) \rangle_{\mathbb{Z}} = \mathrm{Clop}(X)$ and the type of $\pi_X(a_0)$ determines $\alpha(X)$.

Theorem

Let $G = \mathbb{Z}$. Then every G-flow X that has all Sturmian subshifts as factors is of weight at least $2^{\aleph_0}$.

Proof.

Suppose X has all Sturmian subshifts as factors. We may assume that X is zero-dimensional. The Stone duals of the Sturmian subshifts all embed into $\mathrm{Clop}(X)$. Since these Stone duals have elements of $2^{\aleph_0}$ different types, $\mathrm{Clop}(X)$ has to be of size at least $2^{\aleph_0}$. □

Theorem

Let $G = \mathbb{Z}$. Then every G-flow X that has all Sturmian subshifts as factors is of weight at least $2^{\aleph_0}$.

Proof.

Suppose X has all Sturmian subshifts as factors. We may assume that X is zero-dimensional. The Stone duals of the Sturmian subshifts all embed into Clop(X). Since these Stone duals have elements of $2^{\aleph_0}$ different types, Clop(X) has to be of size at least $2^{\aleph_0}$. □

Theorem

Let $G = \mathbb{Z}$. Then every G-flow X that has all Sturmian subshifts as factors is of weight at least $2^{\aleph_0}$.

Proof.

Suppose X has all Sturmian subshifts as factors. We may assume that X is zero-dimensional. The Stone duals of the Sturmian subshifts all embed into $\mathrm{Clop}(X)$. Since these Stone duals have elements of $2^{\aleph_0}$ different types, $\mathrm{Clop}(X)$ has to be of size at least $2^{\aleph_0}$. □

Theorem

Let $G = \mathbb{Z}$. Then every G-flow X that has all Sturmian subshifts as factors is of weight at least $2^{\aleph_0}$.

Proof.

Suppose X has all Sturmian subshifts as factors. We may assume that X is zero-dimensional. The Stone duals of the Sturmian subshifts all embed into $\mathrm{Clop}(X)$. Since these Stone duals have elements of $2^{\aleph_0}$ different types, $\mathrm{Clop}(X)$ has to be of size at least $2^{\aleph_0}$. □

Theorem

Let $G = \mathbb{Z}$. Then every G-flow X that has all Sturmian subshifts as factors is of weight at least $2^{\aleph_0}$.

Proof.

Suppose X has all Sturmian subshifts as factors. We may assume that X is zero-dimensional. The Stone duals of the Sturmian subshifts all embed into $\mathrm{Clop}(X)$. Since these Stone duals have elements of $2^{\aleph_0}$ different types, $\mathrm{Clop}(X)$ has to be of size at least $2^{\aleph_0}$. □

Using a result by Gao, Jackson, and Seward, we can generalize this to all discrete, countably infinite groups.

Theorem (GJS 2009, 2016)

Let G be a discrete, countably infinite group. Then 2^G has $2^{\aleph_0}$ pairwise disjoint, minimal subshifts.

Corollary

For every countably infinite discrete group G, every G-flow X that has all minimal metric G-flows as factors is of weight at least $2^{\aleph_0}$. In particular, there are no universal metric or minimal metric G-flows.

Using a result by Gao, Jackson, and Seward, we can generalize this to all discrete, countably infinite groups.

Theorem (GJS 2009, 2016)

Let G be a discrete, countably infinite group. Then 2^G has $2^{\aleph_0}$ pairwise disjoint, minimal subshifts.

Corollary

For every countably infinite discrete group G, every G-flow X that has all minimal metric G-flows as factors is of weight at least $2^{\aleph_0}$. In particular, there are no universal metric or minimal metric G-flows.

Using a result by Gao, Jackson, and Seward, we can generalize this to all discrete, countably infinite groups.

Theorem (GJS 2009, 2016)

Let G be a discrete, countably infinite group. Then 2^G has $2^{\aleph_0}$ pairwise disjoint, minimal subshifts.

Corollary

For every countably infinite discrete group G, every G-flow X that has all minimal metric G-flows as factors is of weight at least $2^{\aleph_0}$. In particular, there are no universal metric or minimal metric G-flows.

Using a result by Gao, Jackson, and Seward, we can generalize this to all discrete, countably infinite groups.

Theorem (GJS 2009, 2016)

Let G be a discrete, countably infinite group. Then 2^G has $2^{\aleph_0}$ pairwise disjoint, minimal subshifts.

Corollary

For every countably infinite discrete group G, every G-flow X that has all minimal metric G-flows as factors is of weight at least $2^{\aleph_0}$.
In particular, there are no universal metric or minimal metric G-flows.

Using a result by Gao, Jackson, and Seward, we can generalize this to all discrete, countably infinite groups.

Theorem (GJS 2009, 2016)

Let G be a discrete, countably infinite group. Then 2^G has $2^{\aleph_0}$ pairwise disjoint, minimal subshifts.

Corollary

For every countably infinite discrete group G, every G-flow X that has all minimal metric G-flows as factors is of weight at least $2^{\aleph_0}$. In particular, there are no universal metric or minimal metric G-flows.

Thank you for your attention!