University of Messina

On some relative versions of Menger and Hurewicz properties

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Joint work with M. Bonanzinga and F. Maesano

Definition

A space X is

- Menger, briefly M, if for each sequence $(\mathcal{U}_n : n \in \omega)$ of open covers of X there exists a sequence $(\mathcal{V}_n : n \in \omega)$ such that $\mathcal{V}_n$, $n \in \omega$, is a finite subset of $\mathcal{U}_n$ and $X = \bigcup_{n\in\omega} \bigcup \mathcal{V}_n$;

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- Hurewicz, briefly H, if for each sequence $(\mathcal{U}_n : n \in \omega)$ of open covers of X there exists a sequence $(\mathcal{V}_n : n \in \omega)$ such that $\mathcal{V}_n$, $n \in \omega$, is a finite subset of $\mathcal{U}_n$ and for every $x \in X$, $x \in \bigcup \mathcal{V}_n$ for all but finitely many $n \in \omega$.

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Let $\mathcal{U}$ be a cover of a space X and A be a subset of X; the star of A with respect to $\mathcal{U}$ is the set $st(A,\mathcal{U}) = \bigcup\{U : U \in \mathcal{U} \text{ and } U \cap A \neq \emptyset\}$.

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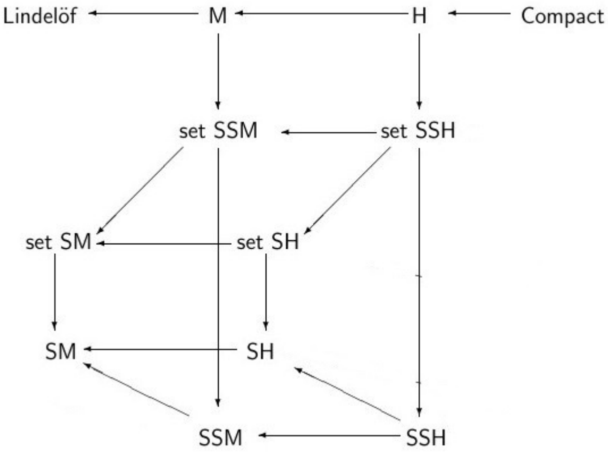
Diagram



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Preliminar results

CC stands for "countably compact"

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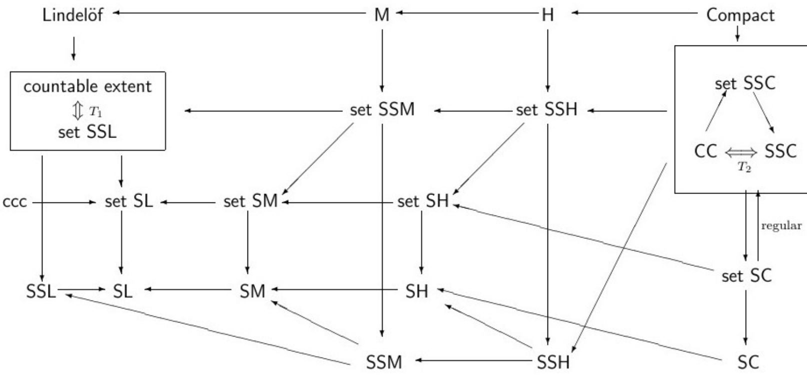
In the class of regular spaces, X CC $\iff$ X set SC

Proposition (B.Mae.; 2020)

In the class of T_1 spaces, X set SSL $\iff$ $e(X) \leq \omega$

Diagram



ccc = "countable chain condition".

Between CC and countable extent

CC $\Rightarrow$ set SSH $\Rightarrow$ set SSM $\Rightarrow$ countable extent.

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$$\text{CC} \Rightarrow \text{set SSH} \Rightarrow \text{set SSM} \Rightarrow \text{countable extent.}$$

Example (B.G.M.; 2022)

A Tychonoff space of cardinality $\mathfrak{d}$ having countable extent which is not set SSM.

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- Let X be a cofinal subset of the Baire space ω^ω of cardinality $\mathfrak{d}$.

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- X is not set SSM: indeed, X is not M and in the class of paracompact Hausdorff spaces we have that M $\Leftrightarrow$ SM.

Where $\mathfrak{d} = \min\{|X| : X \text{ is a cofinal subset of } \omega^\omega\}$

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Corollary (B.G.M.; 2022)

If X is T_1 and $|X| < \mathfrak{d}$, then X set SSM $\iff$ $e(X) \leq \omega$.

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Example (B.G.M.; 2022)

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- The discrete space ω.

The extent of set SM

Theorem (Sakai; 2014)

If X is a regular SM space such that $w(X) = \mathfrak{c}$, then every closed and discrete subspace of X has cardinality less than $\mathfrak{c}$. Hence, we have $e(X) \leq \mathfrak{c}$.

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Set SM vs SM and set SH vs SH

Example (B.G.M.; 2022)

A Tychonoff SC (hence SH and SM) space which is not set SM (hence not set SH and neither set SC).

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- Let $X(\mathfrak{c}) = (2^{\mathfrak{c}} \times \mathfrak{c}^+) \cup (Z \times \{\mathfrak{c}^+\}) \subset 2^{\mathfrak{c}} \times (\mathfrak{c}^+ + 1)$, where Z denotes the set of the points in $2^{\mathfrak{c}}$ with the only the αth coordinate equal to 1.

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- $X(\mathfrak{c})$ is SC [Sakai, 2014], hence SM.

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- $X(\mathfrak{c})$ is SC [Sakai, 2014], hence SM.
- $X(\mathfrak{c})$ it is not set SM. Indeed, $Z \times \{\mathfrak{c}^{+}\}$ is a closed discrete subspace of $X(\mathfrak{c})$ of cardinality $\mathfrak{c}$ and the previous theorem holds.

Set SSM vs set SM

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- $\Psi(\mathcal{A})$ is not set SSM since $e(\Psi(\mathcal{A})) > \omega$.

Set SSH vs set SH

Proposition (B.G.M.; 2022)

If $|X| < \mathfrak{b}$, *then* X *set SL (set SSL)* $\implies$ X *set SH (set SSH).*

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- $\Psi(\mathcal{A})$ is not set SSH

Set SSH vs set SH

Proposition (B.G.M.; 2022)

If $|X| < \mathfrak{b}$, then X set SL (set SSL) $\implies$ X set SH (set SSH).

Question (B.G.M.; 2022)

Does there exist a Tychonoff set SH space which is not set SSH?

Example (B.G.M.; 2022)

($\omega_1 < \mathfrak{b}$) A Tychonoff set SH space which is not set SSH.

- Assume $\omega_1 < \mathfrak{b}$ and consider $\Psi(\mathcal{A}) = \omega \cup \mathcal{A}$ with $|\mathcal{A}| = \omega_1$.
- $\Psi(\mathcal{A})$ is set SH. Indeed, $\Psi(\mathcal{A})$ is separable, hence set SL and by the above proposition.
- $\Psi(\mathcal{A})$ is not set SSH since $e(\Psi(\mathcal{A})) > \omega$.

The product with a compact space

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  - Let Y be a compact space with $c(Y) > \omega$.
  - $X \times Y$ is not SSL, hence not SSM [Bonanzinga, Matveev; 2001].

The product with a compact space

Question (Kočinac, Konca, Singh; 2022)

Is the product of a set SM (set SSM) space with a compact space a set SM (set SSM) space?

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- Partial answer, for set SSM
- NO, for set SM spaces.

On the product of a set SSM space with a compact space

A map is perfect if it is continuous, closed, surjective and each fiber is compact.

Lemma (B.G.M.; 2022)

Uncountable closed discrete subspaces are preserved by perfect maps.

Proposition (B.G.M.; 2022)

The product of a space having countable extent with a compact space has countable extent.

On the product of a set SSM space with a compact space

Proposition (B.G.M.; 2022)

The product of a T_1 set SSL space with a T_1 compact space is set SSL.

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The T_1 product of cardinality less than $\mathfrak{d}$ of a set SSM space with a compact space is set SSM.

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Proposition (B.G.M.; 2022)

If $e(X) > \omega$ and $c(Y) > \omega$, where Y is T_1, then $X \times Y$ is not set SL.

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Example (B.G.M.; 2022)

A set SC (hence set SH, set SM and set SL) space X and a compact space Y such that $X \times Y$ is not set SL (hence neither set SM nor set SH nor set SC).

On the product of a set SM space with a compact space

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- Let $X = \omega_1 \cup A$ be, where $A = \{a_\alpha : \alpha \in \omega_1\}$ is a set of cardinality ω_1; ω_1 has the usual order topology and is an open subspace of X; a basic neighborhood of a point $a_\alpha \in A$ takes the form $O_\beta(a_\alpha) = \{a_\alpha\} \cup (\beta, \omega_1)$, where $\beta < \omega_1$.

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- Y is any compact space with $c(Y) > \omega$.
- X is set SC [Bonanzinga, Maesano, 2020], hence set SM .

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On the product of set SSH spaces with a compact space

Proposition (B.G.M.; 2022)

The product of a set SSH space with a compact space has countable extent.

On the product of set SSH spaces with a compact space

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The product of a set SSH space with a compact space has countable extent.

Proposition (B.G.M.; 2022)

The T_1 product of cardinality less than $\mathfrak{b}$ of a set SSH space with a compact space is set SSH.

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The product of a set SSH space with a compact space has countable extent.

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Question (B.G.M.; 2022)

Is the product of a set SSH space with a compact space a set SSH space?

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Thanks for the attention!