Todorčević' Trichotomy and a hierarchy in the class of tame dynamical systems

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Definitions

A **topological dynamical system** is a pair (X, G), where X is a compact Hausdorff space, and G a topological group which acts on X as a group of homeomorphisms. Thus the G-action is given by a continuous homomorphism $j : G \to \mathrm{Homeo}\,(X), \ j(g) = \breve{g}$, where $\mathrm{Homeo}\,(X)$ is equipped with the uniform topology. Usually, we identify g with $\breve{g}$ and write gx for $\breve{g}x$.
When the acting group is the group of integers $\mathbb{Z}$, the system is often called a **cascade** and we write (X, T) instead of $(X, \mathbb{Z})$; where T is the homeomorphism $j(1)$.

The system (X, G) is :

• **point transitive** if there is a point $x \in X$ whose **orbit** Gx is dense ($\overline{Gx} = X$).

• **minimal** when Gx is dense in X for every $x \in X$.

• **proximal** if for every $x, y \in X$ there is $z \in X$ and a net $g_i \in G$ such that $\lim g_i(x, y) = \lim(g_i x, g_i y) = (z, z)$.

• **strongly proximal** if for every probability measure $\mu \in P(X)$ there is $z \in X$ and a net $g_i \in G$ such that $\lim g_i \mu = \delta_z$.

• A **factor map** or a **homomorphism** $\pi : (X, G) \to (Y, G)$ of dynamical systems is a continuous surjective map such that $\pi(gx) = g\pi(x)$ for all $x \in X$ and $g \in G$. We say that (Y, G) is a **factor** of (X, G) and that (X, G) is an **extension** of (Y, G).
• A **factor map** $\pi : (X, G) \to (Y, G)$ is a **proximal extension** if for every $y \in Y$, every pair of points $x, x' \in X$ with $\pi(x) = \pi(x')$ is proximal. It is **strongly proximal** when for every $y \in Y$ and every probability measure ν with $\mathrm{supp}\ \nu \subset \pi^{-1}(y)$ there is a net $g_i \in G$ and a point $z \in X$ such that $g_i\nu \to \delta_z$.
• It is called an **almost one-to-one extension** if the set $\{x \in X : \iota^{-1}(\iota(x)) = \{x\}\}$ is a dense G_δ subset of X. For a minimal (X, G) this implies that π is a strongly proximal extension.
• Finally an extension π is **isometric** if there is a compatible G invariant “metric” on the subset

$$\{(x, x') \in X \times X : \pi(x) = \pi(x')\}.$$

The enveloping semigroup

The *enveloping semigroup* $E = E(X, G) = E(X)$ of a dynamical system (X, G) is defined as the closure in X^X (with its compact, usually non-metrizable, pointwise convergence topology) of the set $\breve{G} = \{\breve{g} : X \to X\}_{g \in G}$ considered as a subset of X^X. With the operation of composition of maps this is a **right topological semigroup** (i.e. for every $p \in E(X)$ the map $R_p : q \mapsto qp$, $R_p : E(X) \to E(X)$ is continuous).

The BFT dichotomy

The following theorem is due to Bourgain, Fremlin and Talagrand [BFT-78], generalizing a result of Rosenthal.

Theorem (BFT dichotomy)

Let X be a Polish space and let $\{f_n\}_{n=1}^{\infty} \subset C(X)$ be a sequence of real valued functions which is pointwise bounded *(i.e. for each $x \in X$ the sequence $\{f_n(x)\}_{n=1}^{\infty}$ is bounded in $\mathbb{R}$). Let K be the pointwise closure of $\{f_n\}_{n=1}^{\infty}$ in $\mathbb{R}^X$. Then either $K \subset B_1(X)$ (i.e. K is* **Rosenthal compact***) or K contains a homeomorphic copy of $\beta\mathbb{N}$ (the Stone-Čech compactification of a discrete countable set).*

In a seminal paper [Ko-95], Köhler pointed out the relevance of the BFT theorem to the study of enveloping semigroups. She calls a dynamical system, (X, ϕ), where X is a compact Hausdorff space and $\phi : X \to X$ a continuous map, *regular* if for every function $f \in C(X)$ the sequence $\{f \circ \phi^n : n \in \mathbb{N}\}$ does not contain an ℓ_1 sub-sequence (the sequence $\{f_n\}_{n\in\mathbb{N}}$ is an ℓ_1-**sequence** if there are strictly positive constants a and b such that

$$a \sum_{k=1}^{n} |c_k| \le \left\| \sum_{k=1}^{n} c_k f_k \right\| \le b \sum_{k=1}^{n} |c_k|$$

for all $n \in \mathbb{N}$ and $c_1, \ldots, c_n \in \mathbb{R}$). According to BFT "not containing an ℓ_1 sub-sequence" is equivalent to $K \subset B_1(X)$. Since the word "regular" is already overused in topological dynamics I call such systems **tame**.

The corresponding dynamical dychotomy

The following theorem is from [G-06] and [GM-06]:

Theorem (A dynamical BFT dichotomy)

Let (X, G) be a metric dynamical system and let $E(X, G)$ be its enveloping semigroup. We have the following dichotomy. Either

1. *$E(X, G)$ is separable Rosenthal compact, hence with cardinality $\mathrm{card}\, E(X) \leq 2^{\omega}$; or*
2. *the compact space $E(X, G)$ contains a homeomorphic copy of $\beta\mathbb{N}$, hence*
$$\mathrm{card}\, E(X, G) = 2^{2^{\omega}}.$$

A dynamical system is called **tame** if the first alternative occurs, i.e. $E(X, G)$ is Rosenthal compact.

The theorem above can be rephrased as saying that a metric dynamical system (X, G) is either tame (with $\mathrm{card}\, E(X, G) \leq 2^{\omega}$), or $E(X, G)$ contains a topological copy of $\beta\mathbb{N}$. When (X, G) is metrizable and (X, G) is tame then $E = E(X, G)$ is a Fréchet-Urysohn space, and every element $p \in E$ is a limit of a sequence of elements of G,

$$p = \lim_{n\to\infty} g_n.$$

Thus every $p \in E(X, G)$ is a function of **Baire class 1**. (I.e. $f \circ p$ is Baire class 1 for every $f \in C(X)$.)

Examples

Example

Let (X, G) be a point transitive system. Then the action of G on X is **equicontinuous** if and only if $K = E(X, G)$ is a compact topological group whose action on X is jointly continuous and transitive. It then follows that the system (X, G) is isomorphic to the homogeneous system $(K/H, G)$, where H is a closed subgroup of K and G embeds in K as a dense subgroup. When G is Abelian $H = \{e\}$ and $E(X, G) = K$.
A prototypical example of a minimal equicontinuous cascade is an irrational rotation on the circle $(\mathbb{T}, R_\alpha)$.

Example

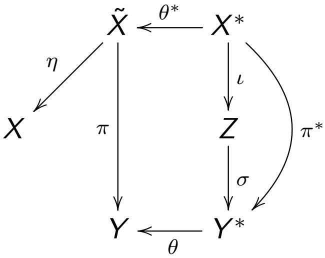
Let G be a discrete group. We form the product space $\Omega = \{0,1\}^G$ and let G act on Ω by translations: $(g\omega)(h) = \omega(g^{-1}h),\ \omega \in \Omega,\ g, h \in G$. The corresponding G-dynamical system (Ω, G) is called the **Bernoulli** G-system. The enveloping semigroup of the Bernoulli system (Ω, G) is isomorphic to the Stone-Čech compactification βG. To see this recall that the collection $\{\bar{A} : A \subset G\}$ is a basis for the topology of βG consisting of clopen sets. Next identify $\Omega = \{0,1\}^G$ with the collection of subsets of G in the obvious way: $A \longleftrightarrow \mathbf{1}_A$. Now define an "action" of βG on Ω by:

$$p * A = \{g \in G : g^{-1}p \in \overline{A^{-1}}\}.$$

It is easy to check that this action extends the action of G on Ω and defines an isomorphism of βG onto $E(\Omega, G)$.

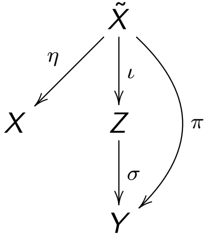
The structure theorem for minimal tame systems

Theorem: [Gl-18] For a general topological group G, a tame, metric, minimal dynamical system (X, G) has the following structure:



Here (i) $\tilde{X}$ is a metric minimal and tame system (ii) η is a strongly proximal extension, (iii) Y is a strongly proximal system, (iv) π is a point distal extension and has a unique RIM (relatively invariant measure), (v) θ, θ^* and ι are almost one-to-one extensions, and (vi) σ is an isometric extension.

When the map π is also open this diagram reduces to



In general the presence of the strongly proximal extension η is unavoidable. If the system (X, G) admits an invariant measure μ then Y is trivial and $X = \tilde{X}$ is an **almost automorphic system**:

$$X \overset{\iota}{\to} Z,$$

where ι is an almost one-to-one extension and Z is equicontinuous. Moreover, μ is unique and ι is a measure theoretical isomorphism $\iota : (X, \mu, G) \to (Z, \lambda, G)$, with λ the Haar measure on Z. Thus, this is always the case when G is amenable.

Corollary

A minimal tame system (X, G) with G amenable has zero topological entropy.

The minimality assumption is superfluous. However, a different kind of machinery is needed in order to prove that fact. This involves combinatorial characterizations of positive entropy (Glasner-Weiss, [GW-95]) and of tameness (Kerr-Li, [KL-07]), which unfortunately, for lack of time, we are not able to describe here.

In a recent work by Fuhrmann, Glasner, Jäger and Oertel [FGJO-21], we show additionally that, with $\iota : X \to Z$ denoting the almost one-to-one extension over the equicontinuous factor, the unique invariant measure μ of X is supported on the dense G_δ subset $X_0 \subset X$, where π is one-to-one:

$$\mu(\{x \in X : \iota^{-1}(\iota(x)) = \{x\}\}) = 1.$$

Such an almost automorphic system is called **regular**.

Some words about the proof of the structure theorem

An essential ingredient of the proof is the following simple fact:

Proposition

Let (X, G) be a metric tame dynamical system. Let $M(X)$ denote the compact convex set of probability measures on X (with the weak* topology). Then each element $p \in E(X, G)$ defines an element $p_* \in E(M(X), G)$ and the map $p \mapsto p_*$ is both a dynamical system and a semigroup isomorphism of $E(X, G)$ onto $E(M(X), G)$.

Proof.

Since $E(X, G)$ is Fréchet we have for every $p \in E$ a sequence $g_i \to p$ of elements of G converging to p. Now for every $f \in C(X)$ and every probability measure $\nu \in M(X)$ we get by the Riesz representation theorem and Lebesgue's dominated convergence theorem

$$g_i\nu(f) = \nu(f \circ g_i) \to \nu(f \circ p) := p_*\nu(f).$$

Since the Baire class 1 function $f \circ p$ is well defined and does not depend upon the choice of the convergent sequence $g_i \to p$, this defines the map $p \mapsto p_*$ uniquely. It is easy to see that this map is an isomorphism of dynamical systems, whence a semigroup isomorphism. As $X \cong \{\delta_x : x \in X\} \subset M(X)$, this map is an injection.

Finally as G is dense in both enveloping semigroups, it follows that this isomorphism is onto. □

On the classification of tame systems

We now assume that our (usually metrizable) dynamical system (X, G) is tame and we ask how complicated is the Fréchet topology on $E(X, G)$. One would expect to see a strong correlation between the complexity of this topology and that of the dynamics of (X, G).

The simplest behavior occurs when $E(X, G)$ is metrizable. Recall that a dynamical system is **non-sensitive** if, for every $\epsilon > 0$ there exists a non-empty open set $O \subset X$ such that for every $g \in G$ the set gO has d-diameter $< \epsilon$. A system (G, X) is **hereditarily non-sensitive** (HNS) if all its subsystems are non-sensitive. The following theorem is by Glasner, Megrelishvili and Uspenskij.

Theorem (GMU-08)

A metrizable dynamical system (X, G) has a metrizable enveloping semigroup iff it is HNS.

A **Sturmian** sequence over $(\mathbb{T}, R_\theta)$. Wikipedia By Siefkenj

Example (Sturmian systems)

The **Sturmian system** is defined as the orbit closure in the Bernoulli system $\Omega = \{0,1\}^{\mathbb{Z}}$ of the Sturmian sequence.

A Sturmian system is tame but not HNS. Its enveloping semigroup is homeomorphic to $\mathbb{Z} \cup DA$, where DA is the **double arrow** space, which is not metrizable.

Note that indeed every Sturmian cascade is an almost one-to-one extension of its largest equicontinuous factor $(\mathbb{T}, R_\alpha)$. However, there are many cascades which have this structure (i.e. they are almost automorphic) which are not even tame; e.g. such a system can have positive entropy.

Example (A generalized Sturmian system)

Let $\alpha = (\alpha_1, \dots, \alpha_d)$ be a vector in $\mathbb{R}^d$, $d \geq 2$ with $1, \alpha_1, \dots, \alpha_d$ independent over $\mathbb{Q}$. Consider the minimal equicontinuous dynamical system (Y, R_α), where $Y = \mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ (the d-torus) and $R_\alpha y = y + \alpha$. Let D be a small closed d-dimensional ball in $\mathbb{T}^d$ and let $C = \partial D$ be its boundary, a $(d-1)$-sphere. Fix $y_0 \in \mathrm{int} D$ and let $X = X(D, y_0)$ be the symbolic system generated by the function $x_0 \in \{0,1\}^{\mathbb{Z}}$ defined by

$$x_0(n) = \chi_D(R_\alpha^n y_0) \quad \text{and set} \quad X = \overline{\mathcal{O}_\sigma x_0} \subset \{0,1\}^{\mathbb{Z}},$$

where σ denotes the shift transformation. It is not hard to check that the system (σ, X) is minimal and admits (Y, R_α) as an almost 1-1 factor:

$$\pi : (X, \sigma) \to (Y, R_\alpha,).$$

Theorem

There exists a ball $D \subset \mathbb{T}^d$ as above such that the corresponding symbolic dynamical system (X, σ) is tame. For such D we then have a precise description of $E(X, \sigma) \setminus \mathbb{Z}$ as the product set $\mathbb{T}^d \times \mathcal{F}$, where $\mathcal{F}$ is the collection of ordered orthonormal bases for $\mathbb{R}^d$.

The following definitions were introduced in [GM-22]:

Definition

Let (X, G) be a tame dynamical system. We say that this system is:

(1) tame$_{\mathbf{1}}$ if $E(X, G)$ is first countable;

(2) tame$_{\mathbf{2}}$ if $E(X, G)$ is hereditarily separable.

The corresponding classes will be denoted by $\mathrm{Tame}_{\mathbf{1}}$ and $\mathrm{Tame}_{\mathbf{2}}$ respectively. It our work we show that the following strict inclusions hold:

$$\mathrm{Equicontinuous} \subset \mathrm{HNS} \subset \mathrm{Tame}_{\mathbf{2}} \subset \mathrm{Tame}_{\mathbf{1}} \subset \mathrm{Tame}.$$

This hierarchy arises naturally from deep results of Todorčević and Argyros–Dodos–Kanellopoulos about separable Rosenthal compacta ([T-99], [ADK-08]).

Theorem (Todorčević' trichotomy)

Let K be a non-metrizable separable Rosenthal compactum. Then K satisfies exactly one of the following alternatives:

(0) *K is not first countable (it then contains a copy of $A(\mathbf{c})$, the Alexandroff compactification of a discrete space of size continuum).*

(1) *K is first countable but K is not hereditarily separable (it then contains either a copy of $D(\{0,1\}^{\mathbb{N}})$, the Alexandroff duplicate of the Cantor set, or $\widehat{D}(S(\{0,1\}^{\mathbb{N}}))$, the extended duplicate of the split Cantor set).*

(2) *K is hereditarily separable and non-metrizable (it then contains a copy of the double arrow).*

Examples

By results of Ellis [Ellis-61], Akin [Akin-98] and [GM-06] we have the following examples:

Examples

(0) The action of the group $G = \mathrm{GL}(d, \mathbb{R})$ on the projective space $\mathbb{P}^{d-1}$, $d \geq 2$, is tame and the corresponding enveloping semigroup $E(\mathbb{P}^{d-1}, G)$ is not first countable (i.e. tame but not tame$_{\mathbf{1}}$).

(1) The action of $G = \mathrm{GL}(d, \mathbb{R})$ on the sphere $\mathbb{S}^{d-1}$ is tame$_{\mathbf{1}}$ but not tame$_{\mathbf{2}}$.

(2) The Sturmian and the generalized Sturmian cascades are tame$_{\mathbf{2}}$ but not HNS.

Some ideas of the proofs

Proposition

Let X be a set, (Y, d) a metric space, and $E \subset Y^X$ a compact subspace in the pointwise convergence topology. The following conditions are equivalent:

1. *A point $p \in E$ admits a countable local basis in E.*
2. *There is a countable set $C \subset X$ which* **determines** p **in** E*, that is, for any $q \in E$, the condition $q(c) = p(c)$ for all $c \in C$ implies that $q(x) = p(x)$ for every $x \in X$.*

Let (X, G) be a dynamical system with enveloping semigroup $E = E(X, G)$. We call an element $p \in E$ a **parabolic idempotent with target** x_0 if there is a point $x_0 \in X$ such that $px = x_0, \forall x \in X$, and a **loxodromic idempotent with target** (x_0, x_1) if there are distinct points $x_0, x_1 \in X$ with $px = x_0, \forall x \in X \setminus \{x_1\}$ and $px_1 = x_1$. We say that x_0 and x_1 are the **attracting** and **repulsing** points of p respectively. Clearly, if (X, G) admits a parabolic idempotent, it is necessarily a proximal system and therefore contains a unique minimal set $Z \subseteq X$. Conversely, if (X, G) is a proximal system then every minimal idempotent is parabolic with target at the unique minimal subset of X.

Proposition

Let (X, G) be a proximal dynamical system. Let $Z \subset X$ be its (necessarily unique) minimal subset.

1. *Suppose that there is an uncountable set $B \subset X$ such that for each $b \in B$ there is a loxodromic idempotent p_b with target (a_b, b), with b as the repulsing point and $a_b \in Z$ the attracting point, such that $b \neq a_b$. Then $E(X, G)$ contains the uncountable discrete subset $\{p_b : b \in B\}$, hence it is not hereditarily separable.*
2. *Suppose there is a point $a \in Z$ and an uncountable set of points $B = \{b_\nu\} \subset X \setminus \{a\}$ such that each pair (a, b_ν) is the target pair of a loxodromic idempotent $p_{(a,b_\nu)}$ with attracting point a and a repulsing point b_ν. Then the parabolic idempotent p_a defined by $p_a x = a$, $\forall x \in X$, does not admit a countable basis for its topology, hence $E(X, G)$ is not first countable.*

Proof.

(1) Straightforward.

(2) Assuming otherwise, in view of the above Lemma, there is a countable set $C \subset X$ such that for any $q \in E(X, G)$, if $qc = p_a c$ for every $c \in C$ then $q = p_a$. Now the set B is uncountable and we can choose an element $b_\nu \in B \setminus C$. It then follows that for every $c \in C$ we have

$$p_{(a,b_\nu)} c = p_a c = a,$$

but nonetheless $p_{(a,b_\nu)} b_\nu = b_\nu \neq a = p_a b_\nu$. Thus $p_{(a,b_\nu)} \neq p_a$, a contradiction. □

Corollary

The action of a **hyperbolic group** *G on its* **Gromov boundary** *∂G is tame but not tame$_{\mathbf{1}}$.*

Example

Example (Dynkin and Maljutov - 1961)

The free group F_2 on two generators, say a and b, is hyperbolic and its boundary can be identified with the compact metric space Ω (a Cantor set) of all the one-sided infinite reduced words w on the symbols a, b, a^{-1}, b^{-1}. The group action is

$$F_2 \times \Omega \to \Omega, \quad (\gamma, w) = \gamma \cdot w,$$

where $\gamma \cdot w$ is obtained by concatenation of γ (written in its reduced form) and w and then performing the needed cancelations. The resulting dynamical system is minimal, strongly proximal, and tame and the enveloping semigroup $E(\Omega, F_2)$ is Fréchet-Urysohn but not first countable.

The β-rank of a tame dynamical system

Let (X, G) be a metric dynamical system, $p \in E(X, G)$ define the the **oscillation function of** p **at** $x \in X$ as

$$\operatorname{osc}(p, x) = \inf\{\sup_{x_1, x_2 \in V} d(px_1, px_2) : V \subset X \text{ open }, x \in V\},$$

and for $A \subset X$ with $x \in A$, $\operatorname{osc}(p, x, A) = \operatorname{osc}(p \restriction A, x)$.
Consider, for each $\epsilon > 0$, the derivative operation

$$A \mapsto A'_{\epsilon,p} = \{x \in A : \operatorname{osc}(p, x, A) \geq \epsilon\}$$

and by iterating define $A^{\alpha}_{\epsilon,p}$ for $\alpha < \omega_1$.

Let

$$\beta(p, \epsilon, A) = \begin{cases} \text{least ordinal } \alpha \text{ with } A^{\alpha}_{\epsilon,p} = \emptyset, & \text{if such an } \alpha \text{ exists} \\ \omega_1 & \text{otherwise.} \end{cases}$$

Set $\beta(p, \epsilon) = \beta(p, \epsilon, X)$ and define the **oscillation rank**

$$\beta(p) = \sup_{\epsilon > 0} \beta(p, \epsilon).$$

Finally define the β-**rank** of the system (X, G) as the ordinal

$$\beta(X, G) = \sup\{\beta(p) : p \in E\}.$$

Via theorems of Bourgain [Bour-80] and Kechris-Louveau [KL-90] we deduce the following:

Theorem

For every metric tame dynamical system (X, T) we have $\beta(X, T) < \omega_1$.

Examples

1. For the Sturmian system we have $\beta(X, T) = 2$.
2. The Dynkin-Maljutov system (Ω, F_2) has β-rank 2.

Question

Is there, for every ordinal $\alpha < \omega_1$, a tame metric system (X, G) with $\beta(X, G) = \alpha$?

Presently I don't even have an example where $\beta(X, G) = 3$ (maybe this is just a good exercise?).

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