A Banach space $C(K)$ reading the dimension of K

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Notation

- K, L - compact Hausdorff spaces,
- $C(K)$ - a Banach space of real-valued continuous functions on K with the norm given by

$$\|f\| = \sup\{|f(x)| : x \in K\},$$

- $X \sim Y$ means that X and Y are isomorphic (not necessarily isometric) Banach spaces,
- $\mathrm{supp}(f) = f^{-1}(\mathbb{R}\setminus\{0\})$ for $f : K \to \mathbb{R}$,
- $\dim K$ - the covering dimension of K.

History

Theorem (Miljutin)

If K, L are uncountable compact metric spaces, then the Banach spaces $C(K)$ and $C(L)$ are isomorphic.

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Theorem (Pełczyński, Semadeni)

If K is a scattered compact space and $C(K) \sim C(L)$, then $\dim L = 0$.

Theorem (Koszmider)

There exists a compact space K such that if $C(K) \sim C(L)$, then L is not zero-dimensional.

Main result

Question

Let $n \in \mathbb{N} \backslash \{0\}$. Is there a compact space K such that $\dim L = n$ whenever $C(K) \sim C(L)$?

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Theorem

Assume $\diamondsuit$. For every $n \in \mathbb{N}$ there is a compact space K_n such that if $C(L) \sim C(K_n)$, then $\dim L = n$.

Spaces with few operators

Definition

We say that a Banach space $C(K)$ has **few operators** if every bounded linear operator $T : C(K) \to C(K)$ is a **weak multiplication** i.e. it satisfies

$$T(f) = fg + S(f)$$

for some continuous function $g \in C(K)$ and weakly compact operator $S : C(K) \to C(K)$.

Theorem (Koszmider, Plebanek)

There exists a connected compact space K such that $C(K)$ has few operators. In such a case $C(K)$ is not isomorphic to any $C(L)$ for L zero-dimensional.

Spaces with few operators

Theorem (Schlackow)

Suppose that K, L are perfect compact spaces, $C(K) \sim C(L)$ and $C(K)$ has few operators. Then K and L are homeomorphic.

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Theorem

Let K be a separable connected compact space such that $C(K)$ has few operators. If $C(K) \sim C(L)$, then K and L are homeomorphic modulo finite set i.e. there are finite subsets $A \subseteq K, B \subseteq L$ such that $K \backslash A$ and $L \backslash B$ are homeomorphic. In particular $\dim K = \dim L$.

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Theorem

Assume $\diamondsuit$. For every $n \in \mathbb{N} \backslash \{0\}$ there is a separable connected compact space K_n such that $\dim K_n = n$ and $C(K_n)$ has few operators.

Spaces with few operators

Definition

A point $x \in K$ is called a **butterfly point**, if there are disjoint open sets $U, V \subseteq K$ such that $\overline{U} \cap \overline{V} = \{x\}$.

Definition

A bounded linear operator $T : C(K) \to C(K)$ is called a **weak multiplier** if $T^*(\mu) = g\mu + S(\mu)$ for some bounded Borel function $g : K \to \mathbb{R}$ and a weakly compact operator $S : C(K)^* \to C(K)^*$.

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Theorem (Koszmider)

Suppose that K is a compact space without butterfly points such that every bounded linear operator $T : C(K) \to C(K)$ is a weak multiplier. Then $C(K)$ has few operators.

Weak multipliers

Lemma (Koszmider)

If a bounded linear operator $T : C(K) \to C(K)$ **is not a weak multiplier**, then there is $\varepsilon > 0$, a sequence $(f_k)_{k\in\mathbb{N}}$ of continuous functions ($f_k : K \to [0,1]$) and a relatively discrete set $\{x_k : k \in \mathbb{N}\} \subseteq K$ such that

- $f_k \cdot f_m = 0$ for $k \neq m$,
- $f_k(x_m) = 0$ for all $k, m \in \mathbb{N}$,
- $|\int f_k d\mu_k| > \varepsilon$ for all $k \in \mathbb{N}$,

where $\mu_k = T^*(\delta_{x_k})$.

Strong extensions

Definition

Let K be a compact space and $(f_k)_{k\in\mathbb{N}}$ be a sequence continuous functions $f_k : K \to [0,1]$ such that $f_k \cdot f_m = 0$ for $k \neq m$. We define the domain of $(f_k)_{k\in\mathbb{N}}$ as

$$D((f_k)_{k\in\mathbb{N}}) = \bigcup\{U : U \text{ is open and } \{k : \mathrm{supp}(f_k) \cap U \neq \varnothing\} \text{ is finite}\}.$$

We say that $L \subseteq K \times [0,1]$ is the **extension** of K by $(f_k)_{k\in\mathbb{N}}$ if and only if L is the closure of the graph of $(\sum_{k\in\mathbb{N}} f_k)|D((f_k)_{k\in\mathbb{N}})$. We say that this is a **strong extension**, if the graph of $\sum_{k\in\mathbb{N}} f_k$ is a subset of L.

Strong extensions

Theorem (Koszmider)

Strong extension of a connected compact space is connected.

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Remark (Barbeiro, Fajardo)

There exists a connected compact space K and its (not strong) extension L, which is not connected.

Idea

We want to construct a compact space $K \subset [0,1]^{\mathfrak{c}}$ as the inverse limit of a sequence $(K_\alpha)_{\alpha<\mathfrak{c}}$, where

- $K_\alpha \subseteq [0,1]^\alpha$,
- $K_{\alpha+1}$ is a strong extension of K_α by some sequence $(f_k^\alpha)_{k\in\mathbb{N}}$,
- if γ is a limit ordinal, then K_γ is the inverse limit of $(K_\alpha)_{\alpha<\gamma}$.

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Moreover, for every $\alpha < \mathfrak{c}$ we are given a sequence of Radon measures $(\mu_k^\alpha)_{k\in\mathbb{N}}$ on $[0,1]^\alpha$ and we require that

- $|\int f_k^\alpha d\mu_k^\alpha| > \varepsilon$ for $k \in \mathbb{N}$ and some $\varepsilon > 0$,

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- $|\int f_k^\alpha d\mu_k^\alpha| > \varepsilon$ for $k \in \mathbb{N}$ and some $\varepsilon > 0$,
- some other technical stuff.

Strong extensions: dimension

Theorem

Let K be a compact metric space with $\dim K = n$. Let L be a strong extension of K by $(f_k)_{k\in\mathbb{N}}$. Suppose that

$$\dim(K \setminus D((f_k)_{k\in\mathbb{N}})) < n.$$

Then

$$\dim L = n.$$

Strong extensions: dimension

Theorem

Let K be an inverse limit of a sequence $(K_\alpha)_{\alpha<\mathfrak{c}}$ such that

- $\dim K_0 = n$,
- $K_{\alpha+1}$ *is a strong extension of* K_α,
- $\dim K_{\alpha+1} = \dim K_\alpha$,
- *if* γ *is a limit ordinal, then* K_γ *is the inverse limit of* $(K_\alpha)_{\alpha<\gamma}$.

Then

$$\dim K = n.$$

Strong extensions: dimension

Suppose that we are at step α in the construction and that K_α satisfies the following condition:

$$\text{For every non-zero Radon measure } \mu \text{ on } K_\alpha \text{ there is a } G_\delta \text{ compact zero-dimensional subset } Z \subseteq K_\alpha \text{ such that } \mu(Z) \neq 0. \quad (*)$$

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Fact

If K_α is a metric compact space of finite dimension, then K_α satisfies $(*)$.

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Fact

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$\diamondsuit \implies CH \implies K_\alpha$ are metrizable for $\alpha < \mathfrak{c}$

Subsets of small dimension with non-zero measure

Problem

Describe the class of finite-dimensional compact spaces K, such that for every non-zero Radon measure μ on K there is a G_δ compact zero-dimensional subset $Z \subset K_\alpha$ such that $\mu(Z) \neq 0$.

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Problem

Describe the class of finite-dimensional compact spaces K, such that for every non-zero Radon measure μ on K there is a G_δ compact zero-dimensional subset $Z \subset K_\alpha$ such that $\mu(Z) \neq 0$.

Question

Is there a finite-dimensional compact space K and a Radon measure μ on K such that $\mu(Z) = 0$ for all zero-dimensional compact $Z \subseteq K$?

Idea

We want to construct a compact space $K \subset [0,1]^{\mathfrak{c}}$ as the inverse limit of a sequence $(K_\alpha)_{\alpha<\mathfrak{c}}$, where

- $K_\alpha \subseteq [0,1]^\alpha$,
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Moreover, for every $\alpha < \mathfrak{c}$ we are given a sequence of Radon measures $(\mu_k^\alpha)_{k\in\mathbb{N}}$ on $[0,1]^\alpha$ and we require that

- $|\int f_k^\alpha d\mu_k^\alpha| > \varepsilon$ for $k \in \mathbb{N}$ and some $\varepsilon > 0$,
- some other technical stuff.

Why $\diamondsuit$?

Assume $\diamondsuit$. Then there is a sequence $(\mu_\alpha)_{\alpha<\omega_1}$ such that

- μ_α is a Radon measure on $[0,1]^\alpha$ for every $\alpha < \omega_1$,
- for every Radon measure μ on $[0,1]^{\omega_1}$ there is a stationary set $S \subseteq \omega_1$ such that
$$\mu | C([0,1]^\alpha) = \mu_\alpha$$
for $\alpha \in S$.

THANK YOU!