On some results about cardinal inequalities for topological spaces

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Hajnal–Juhász' and Arhangel′skiĭ's inequalities

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where $\chi(X)$ is the character and $c(X)$ is the cellularity of X.

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Theorem: [Arhangel′skiĭ's, 1969] If X is a Hausdorff space, then

$$|X| \leq 2^{\chi(X)L(X)},$$

where $L(X)$ is the Lindelöf degree of X.

Pospišil's inequality

The two inequalities are important, in particular, because they show that the two pairs of cardinal functions $L(X)$ and $\chi(X)$, and $c(X)$ and $\chi(X)$, respectively, are sufficient to give an upper bound for the cardinality of a Hausdorff topological space.

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But even Pospišil's inequality from 1937 gives a lower upper bound for the cardinality of a Hausdorff space X than Hajnal–Juhász' and Arhangel′skiĭ's inequalities.

Theorem: [Pospišil, 1937] If X is a Hausdorff space, then

$$|X| \leq d(X)^{\chi(X)},$$

where $d(X)$ is the density and $\chi(X)$ is the character of X.

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Example: If $X = \mathbb{R}$, where $\mathbb{R}$ has the discrete topology, then Hajnal–Juhász' and Arhangel′skiĭ's inequalities give the following estimate: $|X| \leq 2^{\mathfrak{c}\cdot\omega} = 2^{\mathfrak{c}}$,

while Pospišil's inequality gives $|X| \leq \mathfrak{c}^{\omega} = \mathfrak{c}$.

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Theorem: [Bella and Cammaroto, 1988] If X is a Hausdorff space, then

$$|X| \leq d_\theta(X)^{\chi(X)},$$

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Since $d_\theta(X) \leq d(X)$ for every space X, Bella and Cammaroto's inequality is a formal generalization of Pospišil's inequality.

A new generalization of Pospišil's inequality

Theorem: [G–Tkachuk, 2022] If X is a Urysohn space, then

$$|X| \leq d_\theta(X)^{\pi\chi(X)\psi_{\theta^2}(X)},$$

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Therefore Bella and Cammaroto's inequality is equivalent to Pospišil's inequality for Urysohn spaces.

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Therefore G–T inequality mentioned before is not better than Willard and Dissanayake's inequality but it was useful to show that $d(X)^{\chi(X)} = d_\theta(X)^{\chi(X)}$ for every Urysohn space X.

Inequalities with Shanin's number

Theorem: [G–Tkachuk, 2022] If either $\max\{\pi\chi(X), \psi_c(X)\} \geq sh(X)$ or $2^{sh(X)} = sh(X)^+$ for a Hausdorff space X, then $|X| \leq sh(X)^{\pi\chi(X)\cdot\psi_c(X)}$.

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Corollary: Under GCH, if X is a Hausdorff space, then we have the equality $d(X)^{\pi\chi(X)\cdot\psi_c(X)} = sh(X)^{\pi\chi(X)\cdot\psi_c(X)}$.

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Observation: If X is an infinite Hausdorff space, then $d(X)^{\pi\chi(X)\cdot\psi_c(X)} = \pi w(X)^{\pi\chi(X)\cdot\psi_c(X)}$.

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Theorem: [G–Tkachuk, 2022] If either $\max\{\pi\chi(X), t(X)\} \geq sh(X)$ or $2^{sh(X)} = sh(X)^+$ for a regular Hausdorff space X, then $d(X) \leq sh(X)^{\pi\chi(X)\cdot t(X)}$.

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Corollary: Under GCH, if X is a regular Hausdorff space, then we have the inequality $d(X) \leq sh(X)^{\pi\chi(X)}$.

Is $d(X) \leq c(X)^{\pi\chi(X)}$?

In 1978, Fleissner showed that there is a model of ZFC in which GCH holds and there exists a completely regular space X such that $|X| = \omega_2$, $c(X) = \omega_1$ and $\chi(X) = \omega$, and in that way refuting the conjecture that $|X| \leq c(X)^{\chi(X)}$ for every Hausdorff topological space X.

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Therefore, the answer of the above question is negative.

Šapirovskiĭ's inequality

In 1974, Šapirovskiĭ improved Hajnal and Juhász inequality for the class of regular T_1-spaces by replacing $\chi(X)$ with the pseudocharacter $\psi(X)$ and including in the inequality another cardinal function $\pi\chi(X)$ – the π-character of X.

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Theorem: [Šapirovskiĭ, 1974] If X is a regular T_1-space, then $|X| \leq \pi\chi(X)^{c(X)\psi(X)}$.

Notice that Šapirovskiĭ's inequality also overestimates the cardinality of the discrete space $\mathbb{R}$.

Sun's inequality

In 1988, Sun generalized Šapirovskiĭ's and Hajnal and Juhász inequality for the class of all Hausdorff spaces by replacing the pseudocharacter $\psi(X)$ in Šapirovskiĭ's inequality with the closed pseudocharacter $\psi_c(X)$.

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Theorem: [Sun, 1988] If X is a Hausdorff space, then $|X| \leq \pi\chi(X)^{c(X)\psi_c(X)}$.

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Example: If $X = \mathbb{N} \cup \{x\}$, where $x \in \beta\mathbb{N} \setminus \mathbb{N}$, then $|X| = \pi\chi(X) = c(X) = \psi_c(X) = \omega$, $\chi(X) = \mathfrak{c}$ and therefore $\pi\chi(X)^{c(X)\psi_c(X)} = \omega^{\omega\cdot\omega} = 2^\omega = \mathfrak{c}$ while $2^{c(X)\chi(X)} = 2^{\omega\cdot\mathfrak{c}} = 2^{\mathfrak{c}}$.

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Finally, notice that Sun's inequality again overestimates the cardinality of the discrete space $\mathbb{R}$.

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There are example of spaces where $\mathrm{ot}(X) < c(X)$ (e.g. discrete space with cardinality $\mathfrak{c}$) and $\mathrm{ot}(X) < t(X)$ (e.g. the Tychonoff cube $[0,1]^{\mathfrak{c}}$).

G., Tkachenko and Tkachuk's inequalitiy

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Example: If $X = \mathbb{R}$, where $\mathbb{R}$ has the discrete topology, then GTT's inequality gives the following estimate: $|X| \leq \mathfrak{c}^{\omega\cdot\omega} = \mathfrak{c}$, while Sun's inequality, as we have already observed, gives $|X| \leq 2^{\mathfrak{c}}$.

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Therefore, GTT's inequality is strictly stronger than Sun's inequality.

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Therefore, GTT's inequality is strictly stronger than Pospišil's inequality, and therefore, it is stronger also than Arhangel'skiĭ's and Hajnal–Juhász' inequalities.

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Example: [GTT – 2016] For every infinite cardinal κ, there exists a compact Hausdorff space X such that $\mathrm{dot}(X) = \kappa < \min\{\mathrm{ot}(X), \pi\chi(X)\}$. This difference could be arbitrarily large for non-compact spaces.

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Thus, our new inequality improves GTT's and Willard and Dissanayake's inequalities, and therefore it gives either the same or a better upper bound for the cardinality of a Hausdorff space than Sun's, Šapirovskiĭ's, Pospišil's, Hajnal–Juhász' and Arhangel'skiĭ's inequalities mentioned before.

The End

THANK YOU!