Closure spaces, countable conditions and the axiom of choice

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A closure space or a Čech closure space is a pair (X, c) where X is a set and $c : 2^X \to 2^X$ is a closure operator such that:
(i) $c(\emptyset) = \emptyset$; (ii) $A \subseteq c(A)$; (iii) $c(A \cup B) = c(A) \cup c(B)$.
In other words, a Čech closure is a topological (or Kuratowski) closure where the idempotency of the closure is not imposed.
In this talk we will discuss how to transpose to closure spaces some countable notions usual in topological spaces such as: separability, Lindelöfness, first and second countability, ... and study how they compare to each other using the axiom of choice, some weak forms of choice or in a choice-free context.

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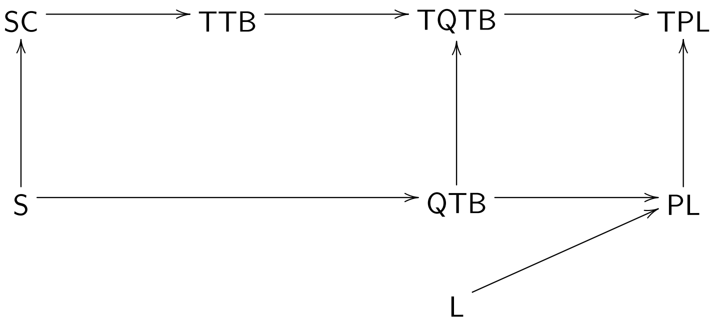
- Topologically preLindelöf (TPL);
- Quasi Totally Bounded (TQTB) – for every $\varepsilon > 0$, exists a countable set $A \subseteq X$ such that $X = \bigcup_{a \in A} B_\varepsilon(a)$;
- Topologically Quasi Totally Bounded(TQTB).



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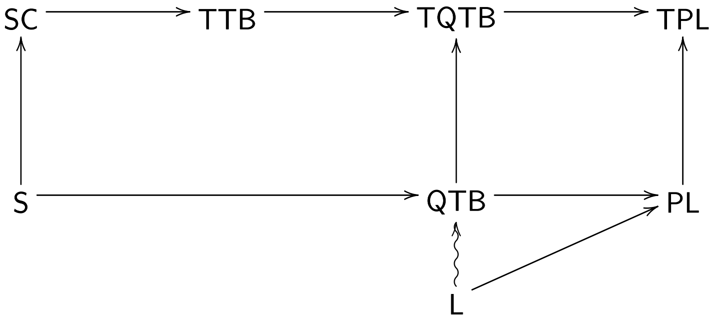

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(v) $d(x, z) \leq d(x, y) + d(y, z)$. [triangle inequality]

Generalized Metric spaces

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A pseudometric on X is a function $d : X \times X \to \mathbb{R}$ such that:

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A quasimetric on X is a function $d : X \times X \to \mathbb{R}$ such that:

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A semimetric on X is a function $d : X \times X \to \mathbb{R}$ such that:

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Every *premetric space* induces a (pre)closure operator.

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Pretopological spaces can equivalently be described with neighborhoods.

$\mathcal{N}_x := \{V \mid x \notin c(X \setminus V)\}$

Neighborhood spaces(=Pretopological spaces)

$$\mathcal{N}: \begin{array}{ccc} X & \longrightarrow & FX, \\ x & \mapsto & \mathcal{N}_x \end{array} \qquad \text{with } FX \text{ the set of filters on } X.$$

$\big(X, (\mathcal{N}_x)_{x \in X}\big)$ is a neighborhood space if for every $V \in \mathcal{N}_x$, $x \in V$.

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$$c(A) = \{x \in X \mid (\forall V \in \mathcal{N}_x)\ V \cap A \neq \emptyset\}$$

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if A is a neighborhood of all its points.

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- A topological space is symmetrizable if it is a semi-metrizable as closure space.

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2. every $\mathcal{W}_x$ is a filter base;
3. $A \subseteq X$ is open if and only if
   for every $x \in A$ there is $W \in \mathcal{W}_x$ such that $x \in W \subseteq A$.

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A topological space X is:

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- *g-second countable* if X has a weak base $(\mathcal{W}_x)_{x \in X}$ such that $\bigcup_{x \in X} \mathcal{W}_x$ is countable.

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Notice that having a countable weak base at each point does not imply being g-first countable.

Compactness in Closure spaces

A closure space (X, c) is *cover-compact* if for every family $\{A_i \mid i \in I\}$ such that $\{c(A_i) \mid i \in I\}$ has the $f.i.p.$, then $\bigcap_{i \in I} c(A_i)$.

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$$\Updownarrow$$

The Kuratowski-Mrówka Theorem is valid, i.e.,

$$(\forall Y)\ p_Y : x \times Y \to Y \text{ is closed.}$$

$$c(p_Y(A) \subseteq p_Y(c(A))$$

Lindelöfness in Closure spaces

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Every cover-Lindelöf space is Lindelöf.

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