Digital-topological k-group structures on digital objects

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Motivations

• Is there a digital image (X, k) with a certain group structure on X ?

• Given a digital image (X, k) with a certain group structure, what relation among elements in the Cartesian product $X \times X$ is the most suitable for establishing a DT-k-group structure on (X, k) ?

Then we are strongly required to have a certain relation making the Cartesian product $X \times X$ connected with respect to the newly-established relation in $X \times X$.

• With a newly-developed adjacency of $X \times X$, say G_*-adjacency. how can we establish a DT-k-group structure on X derived from the given digital image (X, k) ?

Queries

• How to introduce the notion of (G_{k^*}, k_i)-continuity of a map from $(X_1 \times X_2, G_{k^*})$ to (X_i, k_i) ?

• Let $SC_k^{n,l}$ be a simple closed k-curve with l elements in $\mathbb{Z}^n$. Then, how to establish a group structure on $SC_k^{n,l}$? Furthermore, for $SC_k^{n,l}$, we further have the following question.

• How can we formulate a DT-k-group structure from $SC_k^{n,l}$?

Some works related to this talk

[1] S.-E. Han, Digitally topological groups, Electronic Research Archive, Vol.30(6) (2022) 2356-2384.
[2] S.-E. Han, Subgroup structures of DT-k-groups and answers to some open problems, submitted (2022) 1-30.
[3] S.-E. Han, The product property of DT-k-groups, submitted (2022) 1-36.
[4] S.-E. Han, DT-k-rings and DT-k-fields, submitted (2022) 1-23.

Some terminology-1

To develop the notion of a DT-k-group, we will use the following notations.

(1) Dital k-adjacency relations on $\mathbb{Z}^n$:

$$k := k(m,n) = \sum_{i=1}^{m} 2^i C_i^n, \text{where } C_i^n = \frac{n!}{(n-i)!\ i!}. \tag{1.1}$$

For instance,

$$(n,m,k) \in \left\{ \begin{array}{l} (1,1,2), \\ (2,1,4),(2,2,8), \\ (3,1,6),(3,2,18),(3,3,26), \\ (4,1,8),(4,2,32),(4,3,64),(4,4,80), \text{and} \\ (5,1,10),(5,2,50),(5,3,130),(5,4,210),(5,5,242). \end{array} \right\} \tag{1.2}$$

(2) For $X \subset \mathbb{Z}^n, n \in \mathbb{N}$, with a certain k-adjacency of (1.1), we say that the pair (X,k) a digital image.

Some terminology-2

• For a digital image (X, k), two points $x, y \in X$ are k-path connected if there is a finite k-path from x to y in $X \subset \mathbb{Z}^n$. We say that a digital image (X, k) is k-connected (or k-path connected) if any two points $x, y \in X$ is k-path connected. Also, a digital image (X, k) with a singleton is assumed to be k-connected for any k-adjacency.

• A simple closed k-curve with l elements in $\mathbb{Z}^n$, denoted by $SC_k^{n,l}, 4 \leq l \in \mathbb{N}$, is a sequence $(x_i)_{i \in [0, l-1]_{\mathbb{Z}}}$ in $\mathbb{Z}^n$, where x_i and x_j are k-adjacent if and only if $|i - j| = \pm 1 (mod\ l)$. Indeed, the number l of $SC_k^{n,l}$ depends on the situation.

Digital k-neighborhood of the point x_0 and digital (k_0, k_1)-continuity

A. Rosenfeld defined that $f : (X, k_0) \to (Y, k_1)$ is a (k_0, k_1)-continuous map if every k_0-connected subset of (X, k_0) into a k_1-connected subset of (Y, k_1).
In a digital image (X, k), for a point $x_0 \in X$, define a digital k-neighborhood of the point x_0 in (X, k) with radius 1, as follows:

$$N_k(x_0, 1) = \{x \in X \mid x_0 \text{ is } k\text{-adjacenct to } x\}. \qquad (2.4)$$

The digital continuity can be represented by using a digital k-neighborhood in (2.4), as follows:

Proposition 2.1(Han)

Let (X, k_0) and (Y, k_1) be digital images on $\mathbb{Z}^{n_0}$ and $\mathbb{Z}^{n_1}$, respectively. A function $f : X \to Y$ is (k_0, k_1)-continuous if and only if for every $x \in X$, $f(N_{k_0}(x, 1)) \subset N_{k_1}(f(x), 1)$.

G_{k^*}-adjacency relation in $X_1 \times X_2$ which is essential to the establishment of DT-k-group structure

Definition 4.1

Assume two digital images $(X_i, k_i := k_i(t_i, n_i)), X_i \subset \mathbb{Z}^{n_i}, i \in \{1, 2\}$. For distinct points $p := (x_1, x_2), q := (x_1', x_2') \in X_1 \times X_2 \subset \mathbb{Z}^{n_1+n_2}$, we say that the point p is related to q if

$$\left\{\begin{array}{l}(1) \text{ in the case } x_2 = x_2', x_1 \text{ is } k_1\text{-adjacent to } x_1', \text{ and} \\ (2) \text{ in the case } x_1 = x_1', x_2 \text{ is } k_2\text{-adjacent to } x_2'.\end{array}\right\} \quad (4.1)$$

After that, considering this relation within the k^*-adjacency, where $k^* := k(t, n_1 + n_2), t = \max\{t_1, t_2\}$, these two related points p and q are called G_{k^*}-adjacent in $X_1 \times X_2$. Besides, this adjacency is called a G_{k^*}-adjacency of $X_1 \times X_2$ derived from the given two digital images $(X_i, k_i), i \in \{1, 2\}$.

Characterization of the G_{k^*}-adjacency relation in a digital product $X_1 \times X_2$

Remark

(1) In Definition 4.1, we use the notation $(X_1 \times X_2, G_{k^*})$ *to indicate this digital product* $X_1 \times X_2$ *with the* G_{k^*}*-adjacency.*
(2) $(X_1 \times X_2, G_{k^*})$ *is a digital space [1].*
(3) A G_{k^*}*-adjacency relation may not be equal to a* k^**-adjacency one between two points in* $X_1 \times X_2$*. Namely, the* G_{k^*}*-adjacency relation in* $X_1 \times X_2$ *is a new one in* $X_1 \times X_2$ *that need not be equal to a certain k-adjacency relation in* $\mathbb{Z}^{n_1+n_2}$ *of (1.1).*

Characterization of the $(X_1 \times X_2, G_{k^*})$

(1) Two k^*-adjacent points in $X_1 \times X_2 \subset \mathbb{Z}^{n_1+n_2}$ need not be G_{k^*}-adjacent. However, the converse holds. By Definition 4.1, two G_{k^*}-adjacent points in $X_1 \times X_2 \subset \mathbb{Z}^{n_1+n_2}$ are k^*-adjacent.
(2) We strongly stress on the number $k^* := k(t, n_1 + n_2)$ of a G_{k^*}-adjacency relation. Note that the number t is equal to $\max\{t_1, t_2\}$ to determine the number $k^* := k(t, n_1 + n_2)$ for the G_{k^*}-adjacency of $X_1 \times X_2$, where $k_i := k_i(t_i, n_i)), i \in \{1, 2\}$. Namely, the number k^* of a G_{k^*}-adjacency absolutely depends on the given $(X_i, k_i := k_i(t_i, n_i)), i \in \{1, 2\}$ and the number $t = \max\{t_1, t_2\}$.
(3) For instance, consider $SC_8^{2,4} \times SC_8^{2,6}$. Then we have only the G_{32}-adjacency relation in the digital product $SC_8^{2,4} \times SC_8^{2,6}$.

As a special case of Definition 4.1, we define the following:

Definitioin 4.4

Given a digital image $(X, k := k(t, n)), X \subset \mathbb{Z}^n$, the number $k^* := k(t, 2n)$ for a G_{k^*}-adjacency of $X \times X$ is determined by the number t of $(X, k := k(t, n))$ such that any two G_{k^*}-adjacent points in $X \times X$ should only satisfy the condition (4.1) of Definition 4.1.

This G_{k^*}-adjacency of $X \times X$ with the condition of $k^* := k(t, n)$ plays a crucial role in establishing a DT-k-group.

Remark

(1) In Definition 4.4, the number k^* *of* G_{k^*} *is assumed in* $X \times X \subset \mathbb{Z}^{2n}$ *that is different from the number* k *of the* k*-adjacency of the given digital image* $(X, k := k(t, n)), X \subset \mathbb{Z}^n$*.*
(2) In view of Definition 4.4, given $(X, k := k(t, n))$*, there is at least* $k^* := k(t, 2n)$ *establishing a* G_{k^*}*-adjacency of* $X \times X$*.*

Example

(1) Given $(\mathbb{Z}, 2)$, $(\mathbb{Z}^2, G_4)$ exists.
(2) $(SC_k^{n,l} \times [a, b]_{\mathbb{Z}}, G_{k^*})$ exists, where $k^* = k(t, n+1)$ is determined by the number t of $k := k(t, n)$.
(3) For instance, we obtain $(SC_4^{2,8} \times [0, 1]_{\mathbb{Z}})$ with G_6-adjacency (see Figure 1(a)) and $(SC_8^{2,6} \times [0, 1]_{\mathbb{Z}})$ with G_{18}-adjacency (see Figure 1(b)).

Figure 1: Some G_{k^*}-adjacency, $k^* \in \{6, 18\}$

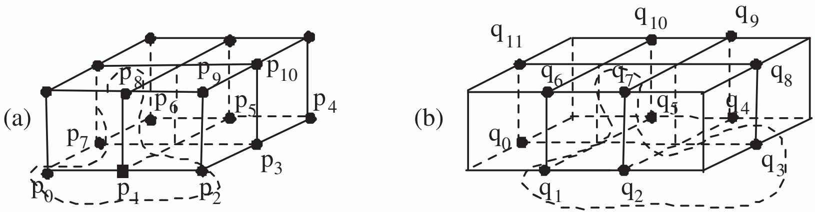


Figure: Configuration of the G_6-adjacency of $SC_4^{2,8} \times [0,1]_{\mathbb{Z}}$ and the G_{18}-adjacency $SC_8^{2,6} \times [0,1]_{\mathbb{Z}}$. In (a), each of the points p_0, p_2 and p_8 is G_6-adjacent to the point p_1. In (b), each of the points q_1, q_3 and q_7 is G_{18}-adjacent to the point q_2.

Lemma

Given two $SC_{k_i}^{n_i,l_i}, i \in \{1,2\}$, *there is always a* G_{k^*}*-adjacency of the digital product* $SC_{k_1}^{n_1,l_1} \times SC_{k_2}^{n_2,l_2}$, *where*
$k^* := k(t, n_1 + n_2), t = max\{t_1, t_2\}$ *and* $k_i := k_i(t_i, n_i), i \in \{1,2\}$.

Example

(1) $SC_8^{2,4} \times SC_8^{2,6} \subset \mathbb{Z}^4$ has a G_{32}-adjacency.
(2) $MSC_{18} \times MSC_{18} \subset \mathbb{Z}^6$ has a G_{72}-adjacency (see Figure 2 and (1.2)).

Lemma

Given two $SC_{k_i}^{n_i,l_i}, i \in \{1,2\}$*, there is always a* G_{k^*}*-adjacency of the digital product* $SC_{k_1}^{n_1,l_1} \times SC_{k_2}^{n_2,l_2}$*, where* $k^* := k(t, n_1+n_2), t = max\{t_1, t_2\}$ *and* $k_i := k_i(t_i, n_i), i \in \{1,2\}$*.*

Example

(1) $SC_8^{2,4} \times SC_8^{2,6} \subset \mathbb{Z}^4$ has a G_{32}-adjacency.
(2) $MSC_{18} \times MSC_{18} \subset \mathbb{Z}^6$ has a G_{72}-adjacency (see Figure 2 and (1.2)).

Figure 2: Configuration of MSC_{18}

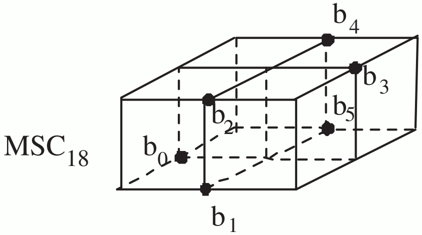


Figure:

Definition 4.9

Given two digital images $(X_i, k_i := k(t_i, n_i)), X_i \subset \mathbb{Z}^{n_i}, i \in \{1, 2\}$, assume the Cartesian product $X_1 \times X_2 \subset \mathbb{Z}^{n_1+n_2}$ with a G_{k^*}-adjacency. For a point $p \in X_1 \times X_2$, we define

$$N_{G_{k^*}}(p) := \{q \in X_1 \times X_2 \,|\, q \text{ is } G_{k^*}\text{-adjacent to } p\} \tag{4.3}$$

and

$$N_{G_{k^*}}(p, 1) := N_{G_{k^*}}(p) \cup \{p\}. \tag{4.4}$$

Then we call $N_{G_{k^*}}(p, 1)$ a G_{k^*}-neighborhood of p.

Corollary

In view of (4.4), for a digital product with a G_{k^*}-adjacency $(X_1 \times X_2, G_{k^*})$ and a point $p := (x_1, x_2) \in X_1 \times X_2$, we have the following:

$$N_{G_{k^*}}(p, 1) = (N_{k_1}(x_1, 1) \times \{x_2\}) \cup (\{x_1\} \times N_{k_2}(x_2, 1)). \tag{4.5}$$

Definition 4.9

Given two digital images $(X_i, k_i := k(t_i, n_i)), X_i \subset \mathbb{Z}^{n_i}, i \in \{1, 2\}$, assume the Cartesian product $X_1 \times X_2 \subset \mathbb{Z}^{n_1+n_2}$ with a G_{k^*}-adjacency. For a point $p \in X_1 \times X_2$, we define

$$N_{G_{k^*}}(p) := \{q \in X_1 \times X_2 \mid q \text{ is } G_{k^*}\text{-adjacent to } p\} \tag{4.3}$$

and

$$N_{G_{k^*}}(p, 1) := N_{G_{k^*}}(p) \cup \{p\}. \tag{4.4}$$

Then we call $N_{G_{k^*}}(p, 1)$ a G_{k^*}-neighborhood of p.

Corollary

In view of (4.4), for a digital product with a G_{k^*}-adjacency $(X_1 \times X_2, G_{k^*})$ and a point $p := (x_1, x_2) \in X_1 \times X_2$, we have the following:

$$N_{G_{k^*}}(p, 1) = (N_{k_1}(x_1, 1) \times \{x_2\}) \cup (\{x_1\} \times N_{k_2}(x_2, 1)). \tag{4.5}$$

(G_{k^*}, k')-continuity of a map

Definition 4.18

Given two digital images (X_i, k_i), $X_i \subset \mathbb{Z}^{n_i}, i \in \{1, 2\}$, consider the digital space $(X_1 \times X_2, G_{k^*})$ and a digital image (Y, k'). A function $f : (X_1 \times X_2, G_{k^*}) \to (Y, k')$ is (G_{k^*}, k')-continuous at a point $p := (x_1, x_2)$ if for any point $q \in X_1 \times X_2$ such that $q \in N_{G_{k^*}}(p)$ (denoted by $p \leftrightarrow_{G_{k^*}} q$), we obtain $f(q) \in N_{k'}(f(p), 1)$ (denoted by $f(p) \Leftrightarrow_{k'} f(q)$). In case the map f is (G_{k^*}, k')-continuous at each point $p \in X_1 \times X_2$, we say that the map f is (G_{k^*}, k')-continuous.

(G_{k^*}, k)-continuity of a map

As a special case of Definitions 4.4 and 4.18, we can consider the following:

Corollary 4.20

Given a digital image $(X, k), X \subset \mathbb{Z}^n$. Consider a Cartesian product $X \times X \subset \mathbb{Z}^{n_1+n_2}$ with a G_{k^*}-adjacency. Consider a map $f : (X \times X, G_{k^*}) \to (X, k)$. For a point $p := (x_1, x_2) \in X \times X$, the map f is (G_{k^*}, k)-continuous at the point p if and only if

$$f(N_{G_{k^*}}(p, 1)) \subset N_k(f(p), 1). \quad (4.10)$$

A map $f : (X \times X, G_{k^*}) \to (X, k)$ is (G_{k^*}, k)-continuous at every point $p \in X \times X$, then the map f is (G_{k^*}, k)-continuous.

Corollary

Let $(X, 2n)$ be a $2n$-connected subset of $(\mathbb{Z}^n, 2n)$. Then each of the typical projection maps $P_i : (X \times X, G_{4n}) \to (X, 2n)$ is a $(G_{4n}, 2n)$-continuous map, $i \in \{1, 2\}$, such that the G_{4n}-adjacency is equal to the typical $4n$-adjacency.

With some hypothesis of the G_{k^*}-adjacency of $X \times X$, the (G_{k^*}, k)-continuity of Corollary 4.20 will play a crucial role in establishing a certain continuity of a multiplication for formulating a DT-k-group (see Definition 5.5, later). Let us compare the (G_{k^*}, k')-continuity and the typical (k, k')-continuity.

Remark (Advantages of the G_{k^*}-adjacency of a digital product and (G_{k^*}, k)-continuity)

Given two digital images (X, k_1) and (Y, k_2), there is always a G_{k^*}-adjacency derived from the two given digital images. Thus the G_{k^*}-adjacency of a digital product will be used in establishing a digital topological version of a typical topological group.

A development of a DT-k-group with the most suitable adjacency for a digital product $X \times X$ from (X, k)

Lemma

The set $\mathbb{Z}^{2n}, n \in \mathbb{N}$, has a G_{4n}-adjacency derived from $(\mathbb{Z}^n, 2n)$ such that this G_{4n}-adjacency is equal to the $4n$-one in $(\mathbb{Z}^n, 2n)$, i.e., $G_{4n} = 4n$.

Let us establish a group structure on the digital image $SC_k^{n,l}$.

Proposition 5.3

Given an $SC_k^{n,l} := (x_i)_{i \in [0,l-1]_{\mathbb{Z}}}$ for any k-adjacency of $\mathbb{Z}^n$, we have a group structure on $SC_k^{n,l}$ with the following operation $*$.

$$* : SC_k^{n,l} \times SC_k^{n,l} \to SC_k^{n,l}$$

given by

$$*(x_i, x_j) = x_i * x_j = x_{i+j(mod\, l)}. \tag{5.1}$$

Then we denote by $(SC_k^{n,l}, *)$ the above group.

Example

(1) Given $SC_k^{n,l} := (x_i)_{i \in [0,l-1]_{\mathbb{Z}}}$ with $l \in \mathbb{N}_0$, there are only two elements such as x_0 and $x_{\frac{l}{2}}$ in $SC_k^{n,l}$ such that $(x_0)^{-1} = x_0$ and $(x_{\frac{l}{2}})^{-1} = x_{\frac{l}{2}}$, where x^{-1} means the inverse element of x (see the two elements x_0, x_3 of $SC_8^{2,6}$).
(2) Given $SC_k^{n,l} := (x_i)_{i \in [0,l-1]_{\mathbb{Z}}}$ with $l \in \mathbb{N}_1$, there is only one element such as x_0 in $SC_k^{n,l}$ whose inverse is itself (see the element x_0 of $SC_{26}^{3,5}$).

Remark

The group $(SC_k^{n,l}, *)$ in Proposition 5.3 is abelian.

Example

(1) Given $SC_k^{n,l} := (x_i)_{i\in[0,l-1]_{\mathbb{Z}}}$ with $l \in \mathbb{N}_0$, there are only two elements such as x_0 and $x_{\frac{l}{2}}$ in $SC_k^{n,l}$ such that $(x_0)^{-1} = x_0$ and $(x_{\frac{l}{2}})^{-1} = x_{\frac{l}{2}}$, where x^{-1} means the inverse element of x (see the two elements x_0, x_3 of $SC_8^{2,6}$).
(2) Given $SC_k^{n,l} := (x_i)_{i\in[0,l-1]_{\mathbb{Z}}}$ with $l \in \mathbb{N}_1$, there is only one element such as x_0 in $SC_k^{n,l}$ whose inverse is itself (see the element x_0 of $SC_{26}^{3,5}$).

Remark

The group $(SC_k^{n,l}, *)$ in Proposition 5.3 is abelian.

Based on the G_{k^*}-adjacency of $X \times X$ and the (G_{k^*}, k)-continuity, we now define the following.

Definition 5.5

A digitally topological k-group, denoted by $(X, k, *)$ and called a DT-k-group for brevity, is a digital image $(X, k := k(t, n))$ combined with a group structure on $X \subset \mathbb{Z}^n$ using a certain binary operation $*$ such that for $(x, y) \in X^2$ the multiplication

$$\alpha : (X^2, G_{k^*}) \to (X, k) \text{ given by } \alpha(x, y) = x * y \text{ is } (G_{k^*}, k)\text{-continuous} \tag{5.3}$$

and the inverse map

$$\beta : (X, k) \to (X, k) \text{ given by } \beta(x) = x^{-1} \text{ is } k\text{-continuous}, \tag{5.4}$$

where the number $k^* := k(t, 2n)$ of the G_{k^*}-adjacency of (5.3) is determined by only the number t of the $k := k(t, n)$-adjacency of the given digital image $(X, k := k(t, n))$.

Remark

In view of Definition 5.5, a DT-k-group, $(X, k, *)$ has the two structures such as the digital image (X, k) and the certain group structure $(X, *)$ satisfying the properties of (5.3) and (5.4).

Theorem 5.8

$(\mathbb{Z}^n, 2n, +)$ is a DT-$2n$-group.

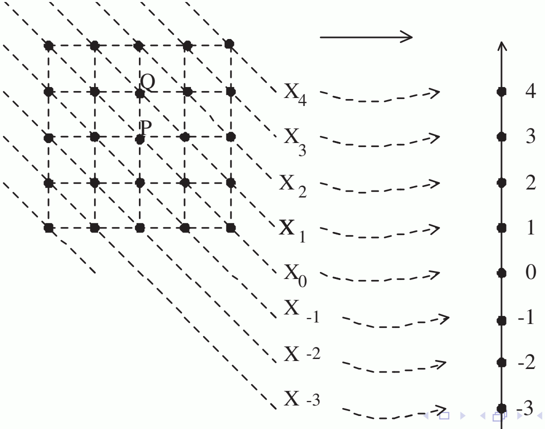
Remark

In view of Definition 5.5, a DT-k-group, $(X, k, *)$ *has the two structures such as the digital image* (X, k) *and the certain group structure* $(X, *)$ *satisfying the properties of (5.3) and (5.4).*

Theorem 5.8

$(\mathbb{Z}^n, 2n, +)$ is a DT-$2n$-group.

Figure.3; The $(G_4, 2)$-continuity of the multiplication from $(\mathbb{Z}^2, G_4) \to (\mathbb{Z}, 2)$ related to being the DT-2-group of $(\mathbb{Z}, 2, +)$, where $P = (0, 0)$ and $Q = (0, 1)$ (see Remark 5.9).



• In a DT-k-group $(X, k, *)$, in case the group $(X, *)$ is abelian, we say that the DT-k-group $(X, k, *)$ is abelian.

• Based on Definition 4.4 and (3.13) and (4.9), Remark 4.25, and Corollary 4.20, let us establish a DT-k-group structure of $(SC_k^{n,l}, *)$ derived from a G_{k^*}-adjacency of the digital product $(SC_k^{n,l} \times SC_k^{n,l}, G_{k^*})$.

Proposition 5.10

$(SC_k^{n,l}, *)$ is a DT-k-abelian group for any k-adjacency of $\mathbb{Z}^n$.

Remark

A finite digital plane $(X, k), X \subset \mathbb{Z}^n$, need not be a DT-k-group.

Example

(1) $(SC_4^{2,4}, *)$ is an abelian DT-4-group.
(2) $(SC_8^{2,6}, *)$ is an abelian DT-8-group.
(3) $(SC_{26}^{3,5}, *)$ is an abelian DT-26-group.
(4) $(MSC_{18}, *)$ is an abelian DT-18-group.

Remark

A finite digital plane $(X, k), X \subset \mathbb{Z}^n$*, need not be a* DT*-k-group.*

Example

(1) $(SC_4^{2,4}, *)$ is an abelian DT-4-group.
(2) $(SC_8^{2,6}, *)$ is an abelian DT-8-group.
(3) $(SC_{26}^{3,5}, *)$ is an abelian DT-26-group.
(4) $(MSC_{18}, *)$ is an abelian DT-18-group.

Ongoing works

As onging works, we can classify DT-k-groups in terms of a certain isomorphism from the viewpoint of DT-k-group theory.

(1) Finding a certain condition supporting the product property of DT-k-group

(2) Establishment of a DT-k-ring and a DT-k-field

(3) Development of a pseudo-DT-k-ring and a pseudo-DT-k-field

(4) Investigation of many examples for DT-k-groups, DT-k-rings, or DT-k-fields

Thanks for your attention!