Ultrafilters and countably compact groups

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Ideals, filters and ultrafilters (on ω)

- $\mathcal{I} \subseteq \mathcal{P}(\omega)$ is an ideal if (1) it is hereditary, (2) closed under finite unions, (3) contains all finite sets, (4) $\omega \notin \mathcal{I}$.
- $\mathcal{F} \subseteq \mathcal{P}(\omega)$ is an filter if (1) it is closed under supersets, (2) closed under finite intersections, (3) contains all co-finite sets, (4) $\emptyset \notin \mathcal{F}$.
- $\mathcal{U} \subseteq \mathcal{P}(\omega)$ is an ultrafilter if it is a maximal (free) filter.
- An ideal $\mathcal{I}$ is tall if every infinite subset of ω contains an infinite set in $\mathcal{I}$.
- Given and ideal $\mathcal{I}$, $\mathcal{I}^+ = \mathcal{P}(\omega) \setminus \mathcal{I}$ denotes the family of $\mathcal{I}$-positive sets.
- Given $\mathcal{X} \subseteq \mathcal{P}(\omega)$,
  $\mathcal{X}^* = \{\omega \setminus X : X \in \mathcal{X}\}$.
- Given $\mathcal{X} \subseteq \mathcal{P}(\omega)$ and $Y \in \mathcal{P}(\omega)$,
  $\mathcal{X} \restriction Y = \{X \cap Y : X \in \mathcal{X}\}$ or $\{X \cap Y : X \in \mathcal{X} \text{ s.t. } |X \cap Y| = \omega\}$.

Brief history of ultrafilters

Henri Cartan (1904-2008)

- H. Cartan. Théorie des filtres. C. R. Acad. Sci. Paris, 205:595–598, 1937.
- H. Cartan. Filtres et ultrafiltres. C. R. Acad. Sci. Paris, 205:777–779, 1937.

Brief history of ultrafilters - pre-history

- F. Riesz, Stetigkeitsbegriff und abstrakte Mengenlehre, Atti IV Congr. Internat. Mat. (Roma 1908), vol. 2, 18-24.

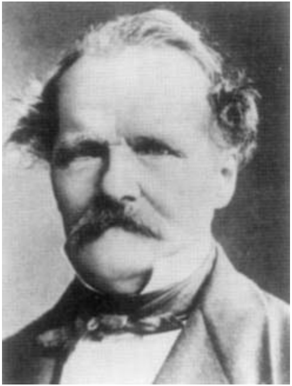
Natur jener Mannigfaltigkeiten anpassen. Eine allgemein anwendbare Definition idealer Verdichtungsstellen ist folgende: Man betrachte als ideale Verdichtungsstelle jedes System von Teilmengen, die folgenden Bedingungen genügt:

1. Jede Obermenge einer Menge des Systems gehört auch dem Systeme an.
2. Wird eine Menge des Systems in zwei Teilmengen zerlegt, so gehört wenigstens eine derselben dem System an.
3. Jedes Paar von Mengen des Systems ist miteinander verkettet.
4. Das System ist vollständig, d. h. es ist in keinem reicheren Systeme, das die Bedingungen 1-3. befriedigt, enthalten.
5. Es gibt kein Element, das Element oder Verdichtungsstelle sämtlicher Mengen des Systems wäre.

Die auf diese Art eigenführten idealen Verdichtungsstellen werden nun als gemeinsame Verdichtungsstelle sämtlicher Mengen des definierenden Systems betrachtet. Es muss dann aber auch für die Verdichtung jener idealen Elemente unter

- L. Vietoris, Stetige Mengen, Monatsh. Math. 31 (1921), 173–204.

Brief history of ultrafilters - ideals

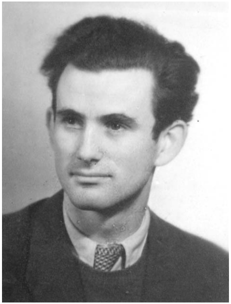
Ernst Eduard Kummer(1810-1893) Julius Wilhelm Richard Dedekind(1831-1916)

- E. E. Kummer . . . ideal numbers (for "unique prime factorization")
- R. Dedekind (1871). . . ideals (in rings).
- M. Stone (1934, 1936). . . (maximal) ideals in Boolean rings, Boolean algebras, . . .

Brief history of ultrafilters - the Polish connection

- S. Ulam, Concerning functions of sets, Fundamenta Mathematicae (1929) Vol. 14, 231–233.
- A. Tarski, Une contribution à la théorie de la mesure, Fundamenta Mathematicae (1930) Vol. 15, 42–50.
- S. Ulam, Zur Masstheorie in der allgemeinen Mengenlehre, Fundamenta Mathematicae (1930) Vol. 16, 140–150.

Brief history of ultrafilters - ultrapowers



Jerzy Łoś (1920-1998)

- T. Skolem, Uber die Nicht-Charakterisierbarkeit der Zahlenreihe mittels endlich oder abzahlbar unendlich vieler Aussagen mit ausschliesslich Zahlenvariablen, Fund. Math. 23 (1934), 150-161.
- E. Hewitt, Rings of real-valued continuous functions, Trans. Amer. Math. Soc. 64 (1948), 45-99.
- J. Łoś, Quelques remarques, théorèmes et problèmes sur les classes définissables d'algèbres. (1955) 98–113. North-Holland Publishing Co., Amsterdam.

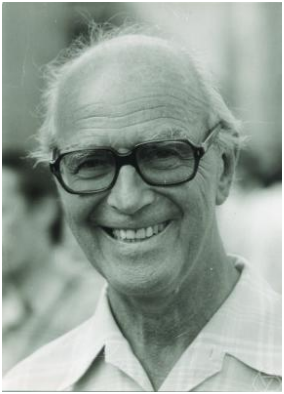
Brief history of ultrafilters - the space of ultrafilters

Eduard Čech (1893-1960)

- M. H. Stone, Applications of the theory of Boolean rings to general topology. Trans. Amer. Math. Soc. 41 (1937), no. 3, 375–481.
- E. Čech, On bicompact spaces. Ann. of Math. (2) 38 (1937), no. 4, 823–844.
- B. Pospíšil, On bicompact spaces. Publ. Fac. Sci. Univ. Masaryk 1939 (1939), no. 270, 16 pp.

Brief history of ultrafilters - there are distinct ultrafilters

Walter Rudin (1921-2010)



Gustave Choquet (1915-2006)

- W. Rudin, Homogeneity problems in the theory of Čech compactifications, Duke Math. J. 23 (1956), 409–419.
- G. Choquet, Construction d'ultrafiltres sur N, Bull. Sci. Math. (2) 92 (1968), 41–48.
- G. Choquet, Deux classes remarquables d'ultrafiltres sur N, Bull. Sci. Math. (2) 92 (1968), 147–153.

Choquet et al - special ultrafilters

Let $\mathcal{U}$ be an ultrafilter on ω. Then $\mathcal{U}$ is

- P-point (δ-stable) (Rudin '56, Gillman-Henriksen '54, Choquet '68)
  $\forall f \in \omega^\omega \; \exists U \in \mathcal{U} \; (f \restriction U$ is constant or fin-to-one),
- Q-point (rare)(Choquet '68)
  $\forall f \in \omega^\omega$ fin-to-one $\exists U \in \mathcal{U} \; (f \restriction U$ is one-to-one),
- selective (absolu) (Choquet '68)
  $\forall f \in \omega^\omega \; \exists U \in \mathcal{U} \; (f \restriction U$ is constant or one-to-one),
- Hausdorff (property C) (Choquet '68)
  $\forall f, g \in \omega^\omega \; \exists U \in \mathcal{U} \; (f \restriction U = g \restriction U$ or $f[U] \cap g[U] = \emptyset)$.

  (A. Connes showed that every selective ultrafilter is property C.)
- rapid (Mokobodzki '69)
  $\forall f \in \omega^\omega \; \exists U = \{k_n : n \in \omega\} \in \mathcal{U} \; \forall n \in \omega \; (f(n) < k_n)$.

Katětov order

Miroslav Katětov (1918-1995)

Definition (Katětov '68).

Let $\mathcal{I}$ and $\mathcal{J}$ be ideals on ω.

- (Katětov order) $\mathcal{I} \leq_K \mathcal{J}$ if there is a function $f : \omega \to \omega$ such that $f^{-1}[I] \in \mathcal{J}$, for all $I \in \mathcal{I}$.
- (Katětov-Blass order) As above with a finite-to-one function f.
- We will say $\mathcal{I}$ and $\mathcal{J}$ are Katětov-equivalent ($\mathcal{I} \simeq_K \mathcal{J}$) if $\mathcal{I} \leq_K \mathcal{J}$ and $\mathcal{J} \leq_K \mathcal{I}$, and analogously for the Katětov-Blass order.

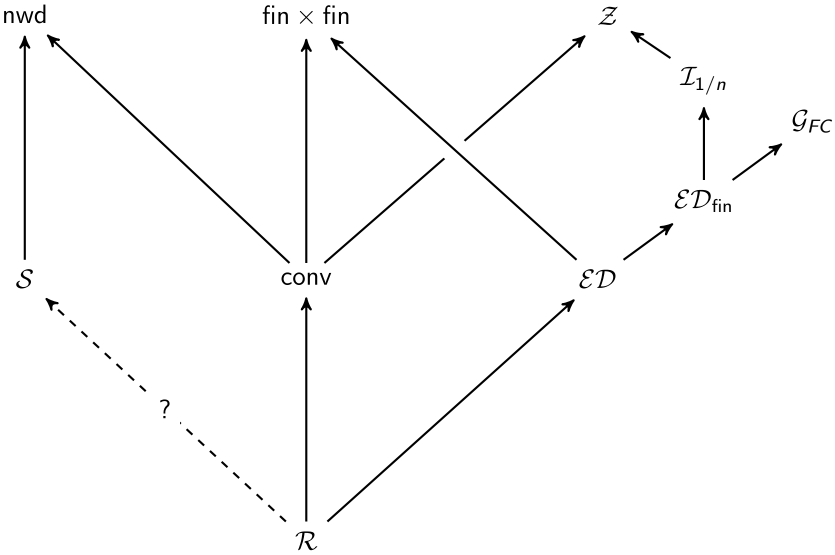
James Earl Baumgartner(1943-2011)

Definition (Baumgartner '95).

Let $\mathcal{U}$ be an ultrafilter and $\mathcal{I}$ an ideal on ω. We say that $\mathcal{U}$ is an $\mathcal{I}$-ultrafilter if $\forall f : \omega \to \omega \ \exists U \in \mathcal{U} \ f[U] \in \mathcal{I}$.

... trivially equivalent to $\mathcal{I} \not\leq_K \mathcal{U}^*$.

The diagram (of some) Borel ideals



Upward cones and classification of ultrafilters

An ultrafilter $\mathcal{U}$ is

- selective iff $\mathcal{ED} \not\leq_K \mathcal{U}^*$ iff $\mathcal{R} \not\leq_K \mathcal{U}^*$
- a P-point iff $\mathit{fin} \times \mathit{fin} \not\leq_K \mathcal{U}^*$ iff $\mathit{conv} \not\leq_K \mathcal{U}^*$
- a Q-point iff $\mathcal{ED}_{\mathit{fin}} \not\leq_{KB} \mathcal{U}^*$,
- rapid iff $\mathcal{I} \not\leq_{KB} \mathcal{U}^*$ for any analytic P-ideal,
- Fubini (or Fatou or property (M)) iff $\mathcal{S} \not\leq_K \mathcal{U}^*$
- Hausdorff iff $\mathcal{G}_{FC} \not\leq_K \mathcal{U}^*$, ...

How trivial can (this) classification of ultrafilters be?

- (Isbell '65) Is it consistent that all ultrafilters have the same Tukey type?
- Is it consistent that all ultrafilters* are in the Katětov order above all Borel ideals?
- (Pospíšil '39) There is an analytic ideal $\mathcal{I}$ and an ultrafilter $\mathcal{U}$ such that $\mathcal{I} \not\leq_K \mathcal{U}^*$.
- (Sakai '18) There is a Borel ideal $\mathcal{I}$ and an ultrafilter $\mathcal{U}$ such that $\mathcal{I} \not\leq_K \mathcal{U}^*$.

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How trivial can this classification of ultrafilters be?

- (Guzmán-H. '20) There is an $F_{\sigma\delta\sigma}$-ideal $\mathcal{I}$ and an ultrafilter $\mathcal{U}$ such that $\mathcal{I} \not\leq_K \mathcal{U}^*$.
- In fact, the $\mathcal{I}$-ultrafilter $\mathcal{U}$ in the statement exists generically, i.e. every filter of character $< \mathfrak{c}$ can be extended to an $\mathcal{I}$-ultrafilter. This result is sharp:
- (Guzmán-H. '20) It is consistent that for every $F_{\sigma\delta}$-ideal $\mathcal{I}$ there is a filter of character $< \mathfrak{c}$ which cannot be extended to an $\mathcal{I}$-ultrafilter.

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Quite trivial!

Theorem (Cancino '21)

It is consistent that $\mathcal{I} \leq_{KB} \mathcal{U}^*$ for every F_σ-ideal $\mathcal{I}$ and ultrafilter $\mathcal{U}$.

In the model:

- There are no Hausdorff ultrafilters.
- There are no Fubini ultrafilters, but
- NCF holds in the model, so there are P-points.
- In fact, the statement follows from NCF +

$$\forall \mathcal{X} \subseteq [\omega]^\omega \ |\mathcal{X}| < \mathfrak{c} \ \forall \mathcal{I} \ F_\sigma \ \exists f : \omega \to \omega \text{ finite-to-one } f[\mathcal{X}] \subseteq \mathcal{I}^+.$$

What about $F_{\sigma\delta}$- ideals?

- Is there an $\mathcal{I}$-ultrafilter for some $F_{\sigma\delta}$ ideal $\mathcal{I}$?
- Is there a $\mathcal{Z}$-ultrafilter?

$$\mathcal{Z} = \left\{ A \subseteq \omega : \lim \frac{|A \cap n|}{n} = 0 \right\}$$

  - (Gryzlov '86) There is (in ZFC) an ultrafilter $\mathcal{U}$ such that for every one-to-one $f : \omega \to \omega$ there is a $U \in \mathcal{U}$ with $f[U] \in \mathcal{Z}$.
  - Assuming $\mathfrak{c} \leq \omega_2$ there is an ultrafilter $\mathcal{U}$ such that for every finite-to-one $f : \omega \to \omega$ there is a $U \in \mathcal{U}$ with $f[U] \in \mathcal{Z}$.
- (Miller '09) Is there a Sacks-indestructible ultrafilter?
  - (Chodounský-Guzmán-H. '21) A Sacks-indestructible ultrafilter is a $\mathcal{Z}$-ultrafilter.

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Versions of topological compactness

Given an ultrafilter p on ω, a sequence $\{x_n : n \in \omega\}$ and a point x contained in a topological space X we say that

$$x = p\text{-}\lim x_n$$

if $\{n \in \omega : x_n \in U\} \in p$ for every neighbourhood U of x.

Definition

Let X be a topological space and let p be a free ultrafilter p on ω.

- X is compact (Alexandroff-Urysohn '29) if every open cover of X has a finite subcover, eqiv. every ultrafilter on X converges.
- X is p-compact (Bernstein '70) if for every sequence $\{x_n : n \in \omega\} \subseteq X$ there is a point $x \in X$ such that $x = p\text{-}\lim x_n$.
- X is countably compact (Fréchet '28) if every countable open cover of X has a finite subcover, eqiv. every sequence has a p-limit for some $p \in \omega^*$,
- X is pseudo-compact (Hewitt '48) if every continuous function $f : X \to \mathbb{R}$ is bounded.

Versions of compactness and products

- (Tychonoff '30/'35) Any product of compact spaces is compact.
- (Bernstein '70) Any product of *p*-compact spaces is *p*-compact.
- (Teresaka '52, Novák '53) There are countably compact spaces whose square is not even pseudo-compact.
- (Comfort-Ross '66) Any product of pseudo-compact topological groups is pseudocompact.

Problem (Comfort '66)

Are there countably compact groups $\mathbb{G}, \mathbb{H}$ such that $\mathbb{G} \times \mathbb{H}$ is not countably compact?

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- (van Douwen '80) Every countably compact group without (non-trivial) convergent sequences contains two countably compact subgroups whose product is not countably compact.
- (Hajnal-Juhász '76) (CH) There is a boolean countably compact group without convergent sequences.

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van Douwen's problem - consistent solutions

Problem (van Douwen '80)

Is there countably compact group without non-trivial convergent sequences?

- (Kuz'minov '58) Every compact topological group contains a convergent sequence.
- (Hajnal-Juhász '76) Yes assuming CH.
- (van Douwen '80) Yes assuming MA.
- (Tomita) Yes assuming MA_{ctble}.
- ...

All of these describe consistent inverse limit constructions of subgroups of $2^{\mathfrak{c}}$.

Main result

Theorem (H.–van Mill–Ramos-García–Shelah 2021)

There is a countably compact subgroup of $2^{\mathfrak{c}}$ without convergent sequences in ZFC.

Iterated ultrapowers of topological spaces

X is dense in $Ult_p(X)$ and
every sequence in X has a p-limit in $Ult_p(X)$: $[f] = p\text{-}\lim f(n)$.

The process can, of course be iterated:

Given a space X, ultrafilter $p \in \omega^*$ and $\alpha < \omega_1$ let

$$Ult_p^{\alpha}(X) = Ult_p(\bigcup_{\beta<\alpha} Ult_p^{\beta}(X))$$

and finally

$$Ult_p^{\omega_1}(X) = \bigcup_{\beta<\omega_1} Ult_p^{\beta}(X).$$

- $Ult_p^{\alpha}(X) = Ult_{p^{\alpha}}(X)$ for $\alpha < \omega_1$,
- $Ult_p^{\omega_1}(X)$ is p-compact, and
- If X is p-compact then $X = Ult_p(X)$.

  (in particular, iterations beyond ω_1 do not produce new spaces).

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The plan

Theorem

There is a countably compact boolean group without convergent sequences.

Find a suitable topological group $\mathbb{G}$ without convergent sequences and consider $Ult_p^{\omega_1}(\mathbb{G})$.

Then $Ult_p^{\omega_1}(\mathbb{G})$ is a p-compact space and a group.

There are two problems to solve:

- Is $Ult_p^{\omega_1}(\mathbb{G})$ with the ultraproduct topology a topological group?
- Does $Ult_p^{\omega_1}(\mathbb{G})$ have convergent sequences?

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Let

$$Hom([\omega]^{<\omega}) = \{\Phi : \ \Phi \text{ is a group homomorphism from } [\omega]^{<\omega} \text{ to } 2\}$$

and let τ_{Hom} be the Bohr topology, i.e. the weakest topology making all $\Phi \in Hom([\omega]^{<\omega})$ continuous.

- $([\omega]^{<\omega}, \tau_{Hom})$ is homeomorphic to a countable dense subgroup of $2^{\mathfrak{c}}$ without convergent sequences.

Claim

$Ult_p^{\omega_1}([\omega]^{<\omega})$ is a p-compact topological group for every p in ω^*.

Every $\Phi \in Hom([\omega]^{<\omega})$ naturally extends to a homomorphism $\overline{\Phi} \in Hom(Ult_p([\omega]^{<\omega}))$ by letting

$$\overline{\Phi}([f]_p) = i \text{ iff } \{k : \Phi(f(k)) = i\} \in p.$$

The ultrapower topology is the topology induced by $\{\bar{\Phi} : \Phi \in Hom([\omega]^{<\omega})\}$ on $Ult_p([\omega]^{<\omega})$.

Similarly, the utrapower topology on $Ult_p^{\omega_1}([\omega]^{<\omega})$ is the topology $\tau_{\overline{Hom}}$ induced by the homomorphisms in $Hom([\omega]^{<\omega})$ extended recursively all the way to $Ult_p^{\omega_1}([\omega]^{<\omega})$ by the same formula as before:

$$\overline{\Phi}([f]) = i \text{ if and only if } \{k : \overline{\Phi}(f(k)) = i\} \in p.$$

The plan works ... for selective ultrafilters

Proposition (H.–van Mill–Ramos-García–Shelah 2021)

p is selective iff for every $\{f_n : n \in \omega\}$ of functions $f_n : \omega \to [\omega]^{<\omega}$ which are not constant or equal on an element of p, there is a sequence $\{U_n : n \in \omega\} \subseteq p$ such that the sequence

$$\{f_n(m) : n \in \omega \text{ and } m \in U_n\}$$

is linearly independent.

Corollary (H.–van Mill–Ramos-García–Shelah 2021)

If p is selective then $Ult_p^{\omega_1}([\omega]^{<\omega})$ is a p-compact topological group without convergent sequences.

Proposition (H.–van Mill–Ramos-García–Shelah 2021)

There is an ultrafilter $p \in \omega^*$ such that $Ult_p^{\omega_1}([\omega]^{<\omega})$ contains a convergent sequence.

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The same ... yet different

Lemma (H.–van Mill–Ramos-García–Shelah 2019)

There is a familly $\{p_\alpha : \alpha < \mathfrak{c}\} \subseteq \omega^*$ such that for every $D \in [\mathfrak{c}]^\omega$ and every sequence $\{f_\alpha : \alpha \in D\} \subseteq ([\omega]^{<\omega})^\omega$ of one-to-one enumerations of linearly independent sets there are $\{U_\alpha : \alpha \in D\}$ such that

1. $\forall \alpha \in D \ U_\alpha \in p_\alpha$, and
2. $\{f_\alpha(n) : \alpha \in D \ \& \ n \in U_\alpha\}$ is a linearly independent subset of $[\omega]^{<\omega}$.

Construct a countably compact topology on $[\mathfrak{c}]^{<\omega}$ starting from $([\omega]^{<\omega}, \tau_{Hom})$ as follows: Fix a family $\{f_\alpha : \omega \leq \alpha < \mathfrak{c}\} \subseteq ([\mathfrak{c}]^{<\omega})^\omega$ s, t.

1. for every infinite $X \subseteq [\mathfrak{c}]^{<\omega}$ there is an $\alpha < \omega_1$ with $rng(f_\alpha) \subseteq X$,
2. Each f_α is a one-to-one enumeration of a linearly independent set, and
3. for every $\alpha < \omega_1$ $rng(f_\alpha) \subseteq [\alpha]^{<\omega}$.

For every $\Phi \in Hom([\omega]^{<\omega})$ define its extension $\overline{\Phi} \in Hom([\mathfrak{c}]^{<\omega})$ recursively by putting

$$\overline{\Phi}(\{\alpha\}) = p_\alpha\text{-}\lim f_\alpha(n).$$

with the topology $\tau_{\overline{Hom}}$ induced by $\{\bar{\Phi} : \Phi \in Hom([\omega]^{<\omega})\}$ on $[\mathfrak{c}]^{<\omega}$.

The same ... yet different

Call a set $D \in [\mathfrak{c}]^\omega$ *suitably closed* if $\omega \subseteq D$ and $\bigcup_{n\in\omega} f_\alpha(n) \subseteq D$ for every $\alpha \in D$.

Proposition

The topology $\tau_{\overline{Hom}}$ contains no non-trivial convergent sequences if and only if
$\forall D \in [\mathfrak{c}]^\omega$ *suitably closed* $\forall X \in [D]^\omega\ \exists \Psi \in Hom([D]^{<\omega})$ *such that*

1. $\forall \alpha \in D\ \Psi(\{\alpha\}) = p_\alpha\text{-}\lim f_\alpha(n)$
2. $|X \cap Ker(\Psi)| = |X \setminus Ker(\Psi)| = \omega$.

Now, if this happens (and it does by our choice of the ultrafilters) then, in particular,

$$K = \bigcap_{\Phi\in Hom([\omega]^{<\omega})} Ker(\overline{\Phi})$$

is finite, and $[\mathfrak{c}]^{<\omega}/K$ with the quotient topology is the Hausdorff countably compact group without non-trivial convergent sequences we want.

Final remarks and questions

Theorem

For every $n \in \omega$ there is a group $\mathbb{G}$ such that $\mathbb{G}^n$ is countably compact while $\mathbb{G}^{n+1}$ is not.

Question

(1) Is there a countably compact group $\mathbb{G}$ without convergent sequences which is not a torsion group, i.e contains a copy of $\mathbb{Z}$?

(2) (Wallace '55) Is every countably compact semigroup with both-sided cancellation a topological group?

Yes to (1) $\Rightarrow$ Yes to (2).

Question

Is it consistent that every p-compact group contains a non-trivil convergent sequence?

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Epilogue - Between Hausdorff and selective

Given an ultrafilter $p \in \omega^*$ consider the following two player game :

I	$\omega = \dot\bigcup_{n\in\omega} I_n, I_0 = I_0^0 \dot\cup I_0^1$		$I_1 = I_1^0 \dot\cup I_1^1$		$\dots$
II		$n_0 \in 2$		$n_1 \in 2$ $\dots$	

Player II wins if $\bigcup_{i\in\omega} I_i^{n_i} \in p$, otherwise Player I wins.

- selective $\Rightarrow$ I has a w.s. $\Rightarrow$ II does not have a w.s. $\Rightarrow$ $\Rightarrow$ $ult_p([\omega]^{<\omega})$ is Hausdorff $\Rightarrow$ p is Hausdorff.
- Assuming CH no arrow reverses,
- In the Cancino model all of the properties are vacuously equivalent.

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Which other configurations are consistent?

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Thank you for your attention!