Ramsey theorem for trees with a successor operation

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Toposym 2022, Prague

Known big Ramsey results by proof techniques

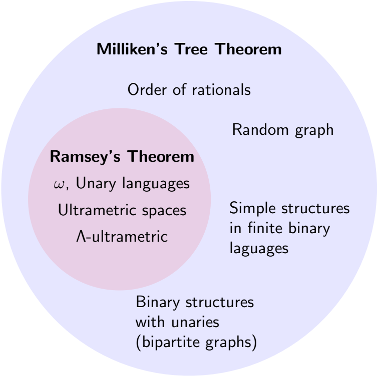
Ramsey's Theorem

ω, Unary languages

Ultrametric spaces

Λ-ultrametric

Known big Ramsey results by proof techniques



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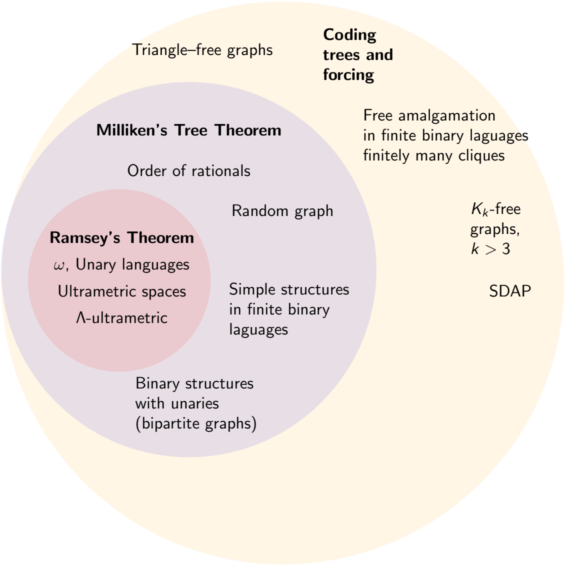

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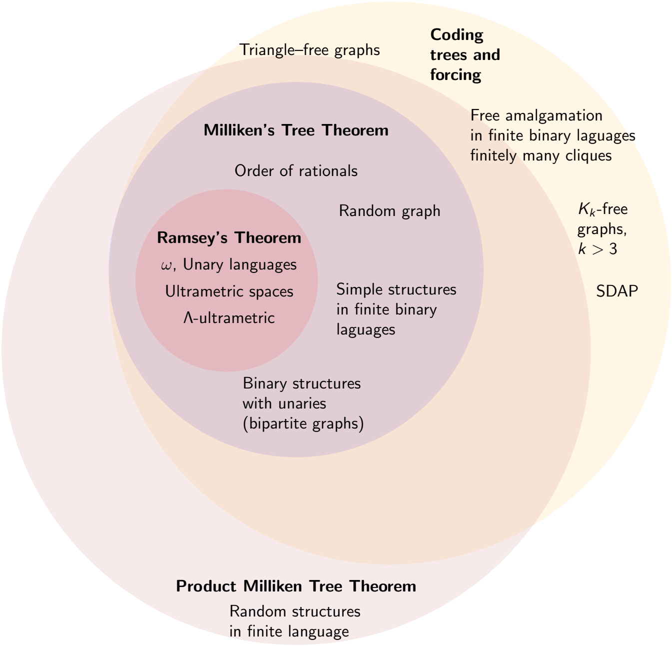

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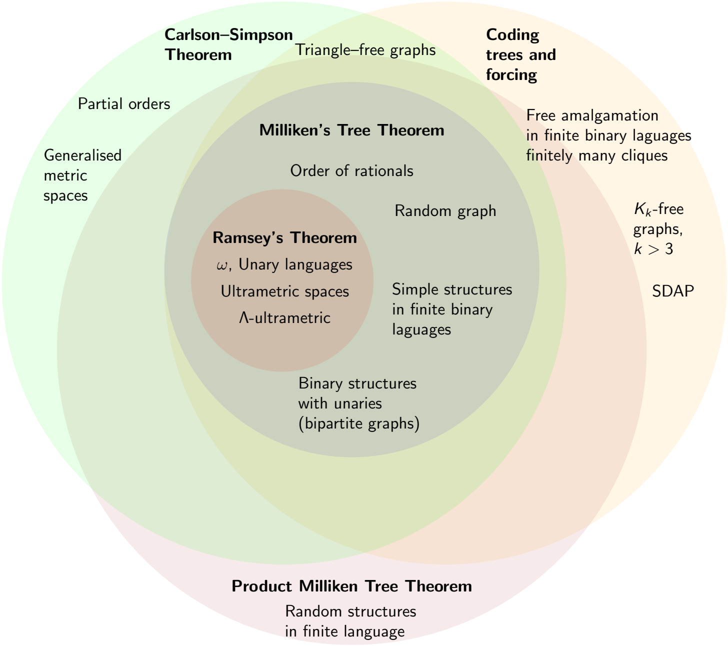
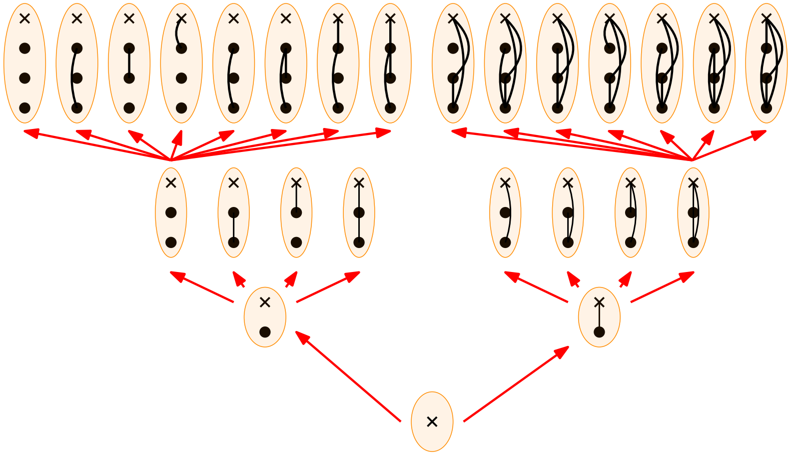


All enumerations tree



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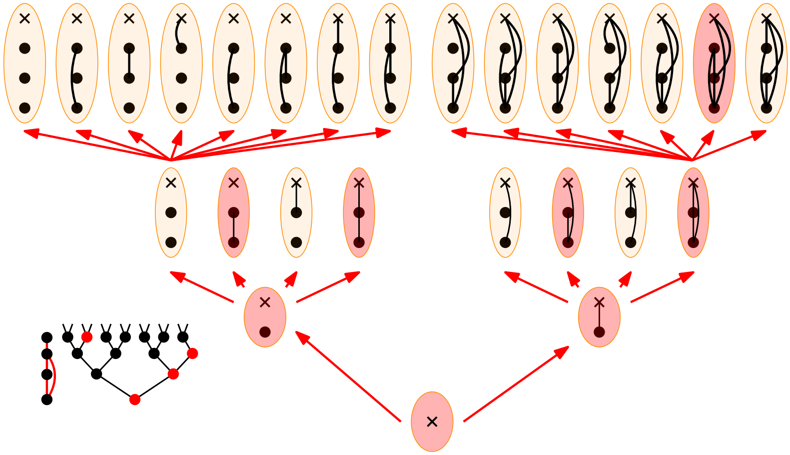

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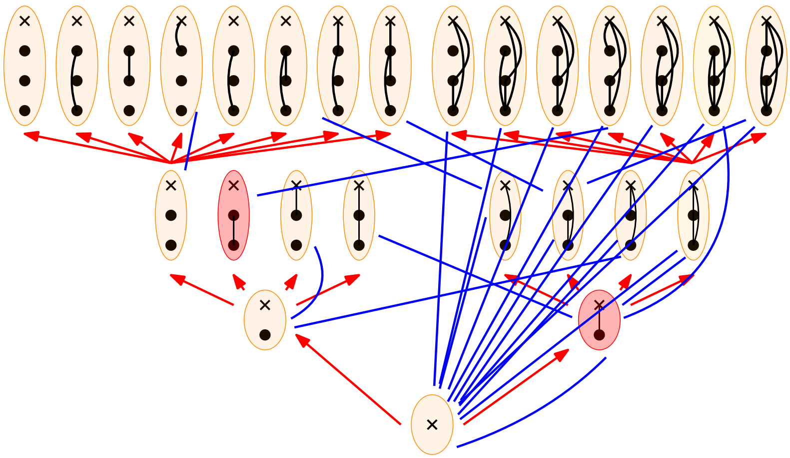

$\mathcal{S}$-trees

A tree is a (possibly empty) partially ordered set $(T, \preceq)$ such that, for every $a \in T$, the set $\{b \in T : b \prec a\}$ is finite and linearly ordered by $\preceq$.
We denote by $\ell(a)$ the level of a and by $a|_n$ the predecessor of a at level n.

Definition ($\mathcal{S}$-tree)

An $\mathcal{S}$-tree is a quadruple $(T, \preceq, \Sigma, \mathcal{S})$ where $(T, \preceq)$ is a countable finitely branching tree with finitely many nodes of level 0, Σ is a set called the alphabet and $\mathcal{S}$ is a partial function $\mathcal{S}\colon T \times T^{<\omega} \times \Sigma \to T$ called the successor operation satisfying the following three axioms:

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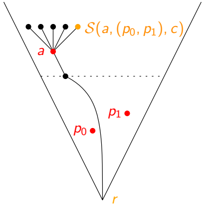

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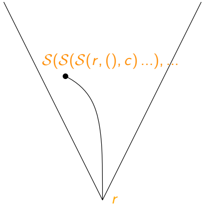
- S3 Constructivity: For every node $a \in T$ of level at least 1, there exist $\bar{p} \in T^{<\omega}$ and $c \in \Sigma$ such that $\mathcal{S}(a|_{\ell(a)-1}, \bar{p}, c) = a$.



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Example

Consider the binary tree of $\{0, 1\}$-words $(B, \sqsubseteq)$ and denote by r its root. $\mathcal{S}$ can be defined only for empty $\bar{p}$ as a concatenation.

$$01011 = \mathcal{S}(\mathcal{S}(\mathcal{S}(\mathcal{S}(\mathcal{S}(r, (), 0), (), 1), (), 0), (), 1), (), 1).$$

Level-decomposition

Definition ($\mathcal{S}$-term)

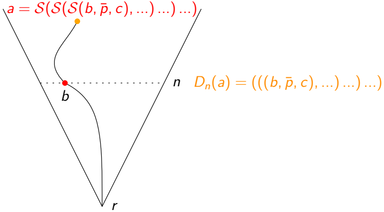
Given an $\mathcal{S}$-tree $(T, \preceq, \Sigma, \mathcal{S})$, we call a term α an $\mathcal{S}$-term if and only if $\alpha \in T$, or $\alpha = (\beta, (\gamma_0, \gamma_1, \dots, \gamma_{n-1}), c)$ where $n \in \omega$, all of $\beta, \gamma_0, \gamma_1 \dots \gamma_{n-1}$ are $\mathcal{S}$-terms and $c \in \Sigma$.

Definition (Level decomposition)

Let $(T, \preceq, \Sigma, \mathcal{S})$ be an $\mathcal{S}$-tree. Given $a \in T$ and $n < \omega$, the level n decomposition of a, denoted by $\mathcal{D}_n(a)$, is an $\mathcal{S}$-term defined recursively:

1. If $\ell(a) \leq n$, then $\mathcal{D}_n(a) = a$.
2. For $a = \mathcal{S}(b, (p_0, \dots, p_{n-1}), c)$ such that $\ell(a) > n$, we let

$$\mathcal{D}_n(a) = (\mathcal{D}_n(b), (\mathcal{D}_n(p_0), \mathcal{D}_n(p_1), \dots, \mathcal{D}_n(p_{n-1})), c).$$



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Example

$$\mathcal{D}_1(001) = ((0, (), 0), (), 1).$$

Manipulating nodes

We denote the class of all $\mathcal{S}$-terms by $\mathcal{T}$. For a set $S \subseteq T$ and a function $f\colon S \to \mathcal{T}$, we denote by $f(\alpha)$ the $\mathcal{S}$-term defined recursively as:

$$f(\alpha) = \begin{cases} f(\alpha) & \text{if } \alpha \in S, \\ \alpha & \text{if } \alpha \in T \setminus S, \\ (f(\beta), (f(\gamma_0), f(\gamma_1), \ldots, f(\gamma_{n-1})), c) & \text{if } \alpha = (\beta, (\gamma_0, \gamma_1, \ldots, \gamma_{n-1}), c). \end{cases}$$

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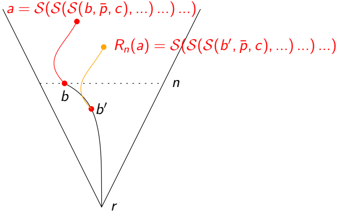
Definition (Level removal)

Given $a \in T$ and $n < \ell(a)$, we let $R_n(a)$ be a node $b \in T$ satisfying $\mathcal{D}_n(b) = r_n(\mathcal{D}_{n+1}(a))$ where r_n is a function $r_n\colon T(n+1) \to \mathcal{T}$ defined by $r_n(d) = d|_n$. If there is no such node b, we say that $R_n(a)$ is undefined.



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Example ($R_1(101) = 11$)

$$\begin{aligned} \mathcal{D}_2(101) &= (10, (), 1), \\ r_1(10) &= 10|_1 = 1, \\ r_1(\mathcal{D}_2(101)) &= r_1((10, (), 1)) = (r_1(10), (), 1) = (1, (), 1) = \mathcal{D}_1(11). \end{aligned}$$

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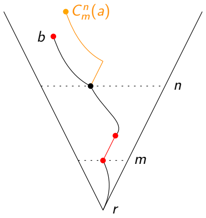
Definition (Level duplication)

Given $a \in T$ and $m < n \leq \ell(a)$, we let $C^n_m(a)$ be a node $b \in T$ satisfying $\mathcal{D}_n(b) = c^n_m(\mathcal{D}_n(a))$ where c^n_m is a function $c^n_m\colon T(n) \to \mathcal{T}$ defined by $c^n_m(d) = (d, \bar{p}, c)$ where $d|_{m+1} = \mathcal{S}(d_m, \bar{p}, c)$. If there is no such node b, we say that $C^n_m(a)$ is undefined.

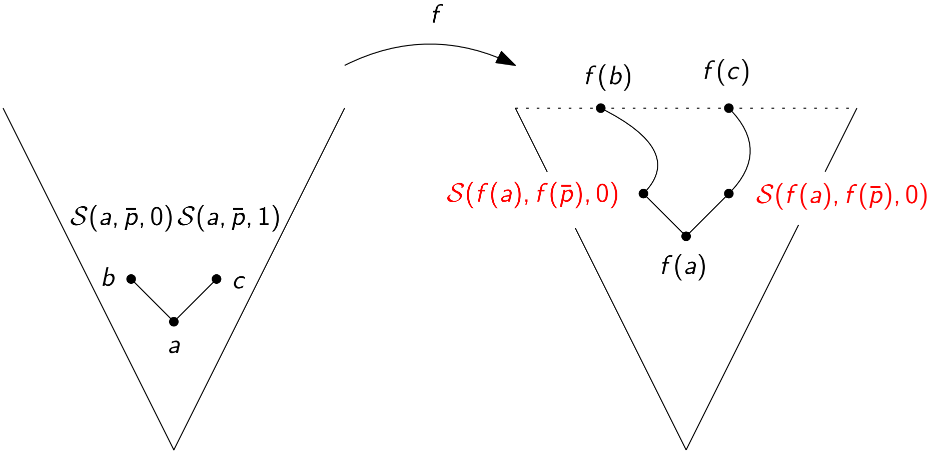


Definition (Shape-preserving functions)

Let $(T, \preceq, \Sigma, \mathcal{S})$ be an $\mathcal{S}$-tree. We call a function $F\colon T \to T$ a shape-preserving function if

1. F is level preserving, and
2. F is weakly $\mathcal{S}$-preserving: If $a = \mathcal{S}(b, \bar{p}, c)$ then $F(a) \preceq \mathcal{S}(F(b), F(\bar{p}), c)$

Function $f\colon S \to T$, $S \subseteq T$ is shape-preserving if it extends to a shape-pres. $F\colon T \to T$.



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Theorem (Balko, Chodounský, Dobrinen, H., Konečný, Nešetřil, Zucker, Vena, 2021+)

Let $(T, \preceq, \Sigma, \mathcal{S})$ be an $\mathcal{S}$-tree. Assume that $\mathcal{S}$ satisfies the following conditions:

- S4 *Level removal: For every $a \in T$, $n < \ell(a)$ such that $\mathcal{D}_{n+1}(a)$ does not use any nodes of level n, the node $R_n(a)$ is defined.*
- S5 *Level duplication: For every $a \in T$, $m < n \leq \ell(a)$, the node $C^n_m(a)$ is defined.*
- S6 *Decomposition: For every $n \in \omega$, $g \in \mathrm{Shape}(T(\leq n), T)$ such that $n > 0$ and $\tilde{g}(n) > \tilde{g}(n-1) + 1$, there exists $g_1 \in \mathrm{Shape}(T(\leq n), T)$ and $g_2 \in \mathrm{Shape}_{\tilde{g}(n)-1}(T(\leq(\tilde{g}(n) - 1), T))$ such that $\tilde{g}_1(n) = \tilde{g}(n) - 1$ and $g_2 \circ g_1 = g$.*

Then, for every $k \in \omega$ and every finite colouring χ of $\mathrm{Shape}(T(\leq k), T)$, there exists $F \in \mathrm{Shape}(T, T)$ such that χ is constant when restricted to $\mathrm{Shape}(T(\leq k), F[T])$.

Ramsey theorem for shape-preserving functions

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Application to free amalgamation classes

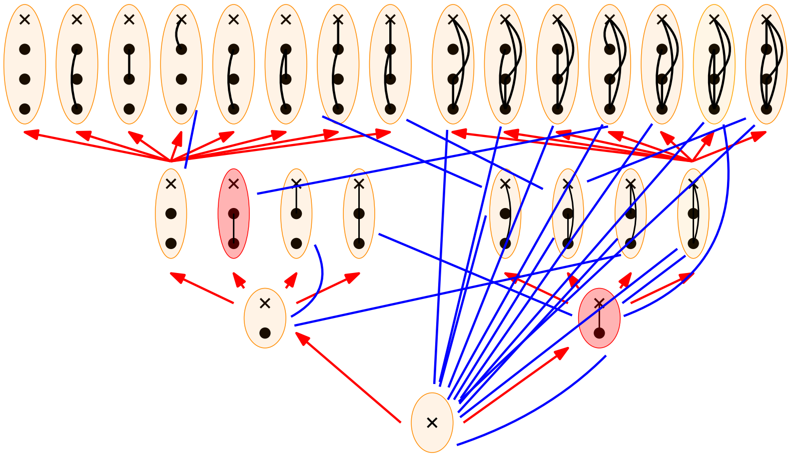
We want to give non-forcing proof of:

Theorem (Zucker 2020+)

Let L be a finite binary language and $\mathcal{F}$ a finite family of irreducible L-structures. Then every countable universal $\mathcal{F}$-free structure has finite big Ramsey degrees.

We fix family $\mathcal{F}$. Examples are for $\mathcal{F} = \{K_4\}$.

All enumerations tree



Constructing all enumeration tree

Definition (Type)

Type of level n is an $\mathcal{F}$-free L-structure $\mathbf{A}$ with vertices $\{0, 1, \ldots, n-1, t\}$, where t is the type vertex.

Definition (Levelled type)

Levelled type of level n is a pair $\mathbf{a} = (\mathbf{A}, \mathrm{fl}_{\mathbf{A}})$ where $\mathbf{A}$ is a type of level n and $\mathrm{fl} : n \setminus \{0\} \to n$ is a function satisfying:

1. $\mathrm{fl}_{\mathbf{a}}(i) < i$.
2. whenever $i < j$ forms an edge of $\mathbf{A}$ then $\mathrm{fl}_{\mathbf{A}}(j) > i$.

Nodes of an $\mathcal{S}$-tree are levelled types ordered by inclusion. Successor operation is an amalgamation.

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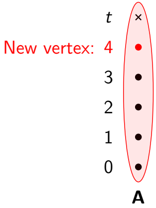

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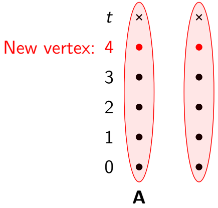

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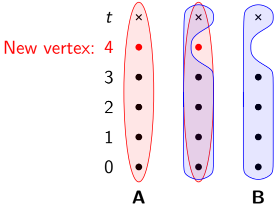
Type of level n is an $\mathcal{F}$-free L-structure $\mathbf{A}$ with vertices $\{0, 1, \ldots, n-1, t\}$, where t is the type vertex.

Definition (Levelled type)

Levelled type of level n is a pair $\mathbf{a} = (\mathbf{A}, \mathrm{fl}_{\mathbf{A}})$ where $\mathbf{A}$ is a type of level n and $\mathrm{fl} : n \setminus \{0\} \to n$ is a function satisfying:

1. $\mathrm{fl}_{\mathbf{a}}(i) < i$.
2. whenever $i < j$ forms an edge of $\mathbf{A}$ then $\mathrm{fl}_{\mathbf{A}}(j) > i$.

Nodes of an $\mathcal{S}$-tree are levelled types ordered by inclusion. Successor operation is an amalgamation.



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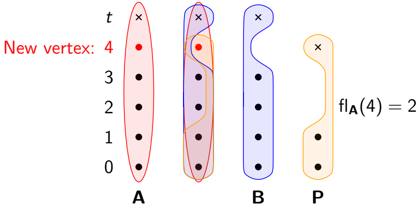

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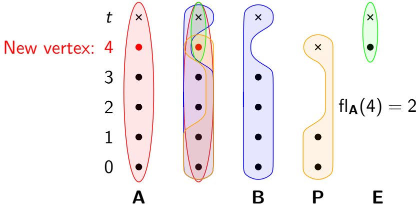

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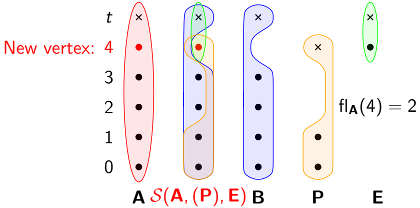

Non-forcing proof of Zucker's theorem

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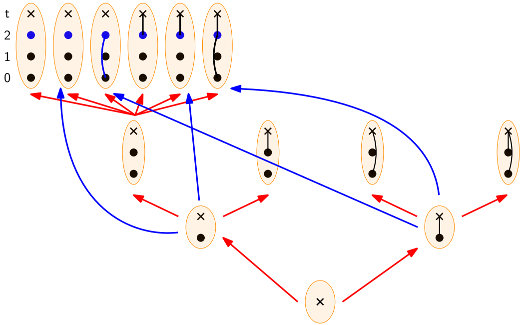

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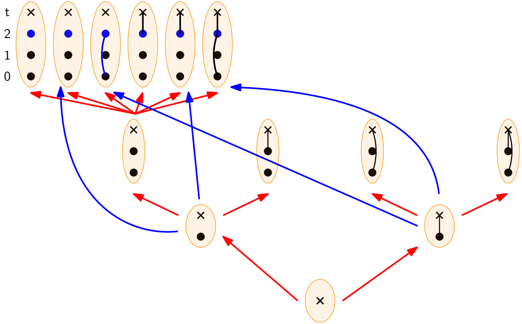

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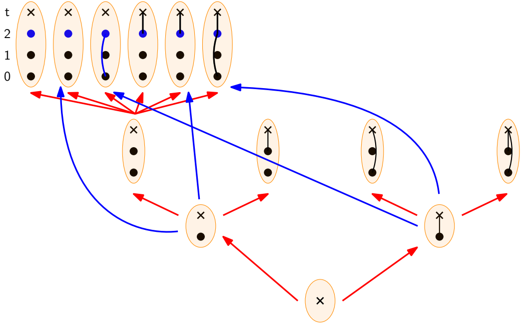

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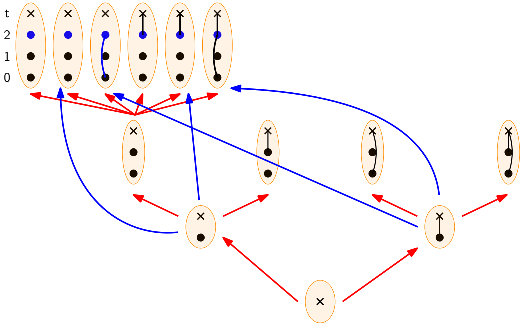

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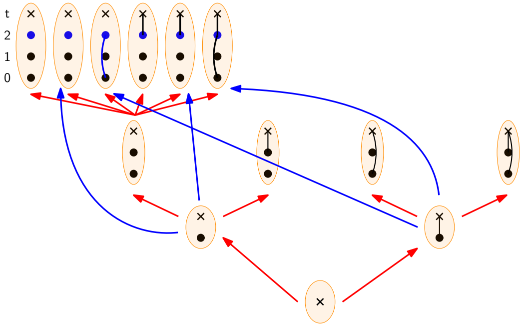

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7. Verify that envelopes are bounded for nice copies inside nice enumerations (same was as in Zucker's paper)

More general result

Theorem

Let L be a finite language consisting of unary and binary symbols, and let $\mathcal{K}$ be a hereditary class of finite structures and $k \geq 2$. Assume that every countable structure $\mathbf{A}$ *has a completion to $\mathcal{K}$ provided that every induced cycle in* $\mathbf{A}$ *(seen as a substructure) has a completion in $\mathcal{K}$ and every irreducible substructure of* $\mathbf{A}$ *of k embeds into $\mathcal{K}$. Then $\mathcal{K}$ has a Fraïssé limit with finite big Ramsey degrees.*

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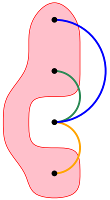
Open problem: Dpes class of all finite structures omitting the following substructure have finite big Ramsey degrees?



Thank you for the attention

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