Periodicity of solenoidal automorphisms

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Terminology

- A dynamical system is a pair (X, f) where X is a topological space and f is a continuous self map on X.
- Given a point $x \in X$, the sequence $(x,\ f(x),\ f^2(x), ...)$ is called the trajectory of x.
- A point $x \in X$ is called a periodic point of f if $f^n(x) = x$ for some $n \in \mathbb{N}$ and the least such n is called the period of x.

Sets of periods/periodic points

The problems of characterizing the sets of periods and periodic points of a family of dynamical systems have been well-studied in the literature.
To put formally, we seek the following:
If $\mathfrak{F}$ is a family of maps on a space X, then give a characterization of the collections:

$$\{Per(f) : f \in \mathfrak{F}\}$$

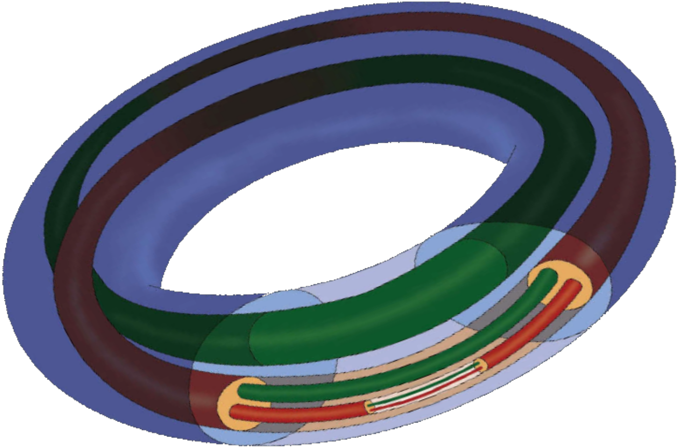
where, $Per(f) = \{n \in \mathbb{N} : f \text{ has a periodic point of period } n\}$,

and

$$\{P(f) : f \in \mathfrak{F}\}$$

where , $P(f) = \{x \in X : x \text{ is a periodic point of } f\}$.

The Dyadic Solenoid



The Dyadic Solenoid

Consider the solid torus :

$$\mathsf{T} = S^1 \times D^2 = \{(\phi, x, y) |\ 0 \leq \phi < 1,\ x^2 + y^2 \leq 1\}.$$

Fix a $\lambda \in \mathbb{R}$ such that $\lambda \in \left(0, \frac{1}{2}\right)$.

Define F: T $\rightarrow$ T such that

$$\mathsf{F}(\phi, x, y) = \left(2\phi \,(mod\, 1)\,, \lambda x + \tfrac{1}{2}\, \cos 2\pi\phi, \lambda y + \tfrac{1}{2}\, \sin 2\pi\phi\right)$$

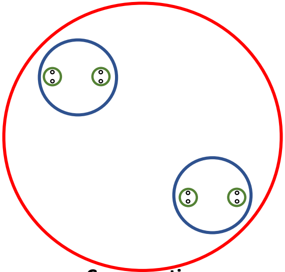
The Dyadic Solenoid

The map F stretches by a factor of 2 in the S^1-direction, contracts by a factor of λ in the D^2-direction.

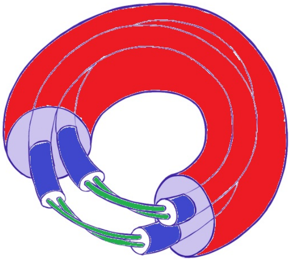
F wraps the image twice inside T and the image $F^{n+1}(T)$ is contained inside $int(F^n(T))$.

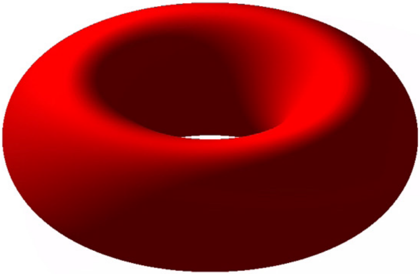
The set $\mathrm{S} = \cap_{n=0}^{\infty} F^n(\mathrm{T})$ is a **dyadic solenoid**.

The Dyadic Solenoid



Cross-section





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Introduction: Solenoid

Definition

A compact connected finite-dimensional abelian group is called a **solenoid**.

Equivalently, a topological group Σ is a solenoid if and only if its Pontryagin dual group $\widehat{\Sigma}$ is (isomorphic to) a subgroup of the discrete additive group, $\mathbb{Q}^n$ and contains $\mathbb{Z}^n$ i.e., $\mathbb{Z}^n \leq \widehat{\Sigma} \leq \mathbb{Q}^n$. In particular, when

- $\widehat{\Sigma} = \mathbb{Z}^n$, Σ is an $n-$dim Torus.
- $\widehat{\Sigma} = \mathbb{Q}^n$, Σ is called a $n-$dim Full Solenoid.
- $\mathbb{Z}^n < \widehat{\Sigma} < \mathbb{Q}^n$, $\Sigma :=$ General Solenoid (n-dim).

Solenoid as an inverse limit

Inverse Limit

Let X_k be a topological space for each $k \in \mathbb{N}_0$ and $f_k : X_k \to X_{k-1}$ be a continuous map for each $k \in \mathbb{N}$. Then the subspace of $\prod_{k=0}^{\infty} X_k$ defined as $\varprojlim_k (X_k, f_k) = \{(x_k) \in \prod_{k=0}^{\infty} X_k : x_{k-1} = f_k(x_k), \forall k \in \mathbb{N}\}$ is called the inverse limit of the sequence of maps (f_k).

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One dimensional solenoid

Let $A = (a_1, a_2, \cdots)$ be a sequence of integers such that $a_k \geq 2$ for every $k \in \mathbb{N}$. The solenoid corresponding to the sequence A, denoted by Σ_A, is defined as
$\Sigma_A = \{(x_k) \in (S^1)^{(\mathbb{N}_0)} : x_{k-1} = a_k x_k \,(mod\, 1) \text{ for every } k \in \mathbb{N}\}$.

Solenoid as an inverse limit

Relation between both descriptions

The dual of a one dimensional solenoid Σ_A, where $A = (a_k)$ is isomorphic to the subgroup of $\mathbb{Q}$ generated by $\{\frac{1}{a_1 a_2 \cdots a_k} : k \in \mathbb{N}\}$.

Height Sequences

Let $S \subset \mathbb{Q}$ and $x \in S$. For a $p \in P$, the $p-$height of x with respect to S, denoted by $h_p^{(S)}(x)$ is defined as the largest non-negative integer n, if it exists, such that $\frac{x}{p^n} \in S$; otherwise, define $h_p^{(S)}(x) = \infty$. Thus, we have a sequence $(h_p^{(S)}(x))$, p ranging over prime numbers in the usual order, with values in $\mathbb{N}_0 \cup \{\infty\}$. We call such sequences as *height sequences*.

Solenoid as an inverse limit

Height Sequences

If (u_p) and (v_p) are two height sequences such that $u_p = v_p$ for all but finitely many primes and $u_p = \infty \Leftrightarrow v_p = \infty$, then they are said to be equivalent. If S is a subgroup of $\mathbb{Q}$, then there is a unique height sequence (up to equivalence) associated to all non-zero elements of S. Also, two subgroups of $\mathbb{Q}$ are isomorphic if and only if their associated height sequences are equivalent.

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Terminology

The field of $p-$adic numbers, $\mathbb{Q}_p$ is the completion of $\mathbb{Q}$ under the $p-$adic norm, defined by $|\frac{a}{b}|_p = \frac{1}{p^n}$, where n is the integer such that $\frac{a}{b} = p^n \frac{a'}{b'}$ and p divides neither a' nor b'.Let $\mathbb{Z}_p = \{x \in \mathbb{Q}_p : |x|_p \leq 1\}$ denote the ring of $p-$adic integers. Then define $n_p^{(S)} = sup\{h_p^{(S)}(x) : x \in S \cap \mathbb{Z}_p^*\}$, where $\mathbb{Z}_p^*$ is the multiplicative group $\{x \in \mathbb{Z}_p : |x|_p = 1\}$.

Group of adeles

The group of adeles $\mathbb{Q}_{\mathbb{A}}$ is defined as the restricted product

$$\mathbb{R} \times \prod_{p \in P} \mathbb{Q}_p \text{ with respect to } \mathbb{Z}_p$$

i.e., for any $(a_\infty, a_2, a_3, \ldots) \in \mathbb{Q}_{\mathbb{A}}$, $a_p \in \mathbb{Z}_p$ for all but finitely many p.

Note: Since every rational number has $p-$adic norm equal to 1 for all but finitely many p, we have a diagonal inclusion $\delta : \mathbb{Q} \to \mathbb{Q}_{\mathbb{A}}$ given by $(\delta(r))_p = r$ for every $p \leq \infty$ and for every $r \in \mathbb{Q}$.

Dual group of $\mathbb{Q}$

For any $a = (a_p) \in \mathbb{Q}_{\mathbb{A}}$, we can associate a character ψ_a of $\mathbb{Q}$ as

$$\psi_a(r) = e^{-2\pi i r a_\infty} \prod_{p<\infty} e^{2\pi i \{r a_p\}_p},$$

where $\{x\}_p$ is the $p-$adic fractional part of x (i.e., the sum of the terms "with" negative power of p in the p-adic expansion of x).
The map $\psi : \mathbb{Q}_{\mathbb{A}} \to \widehat{\mathbb{Q}}$ given by

$$a \mapsto \psi_a$$

is a surjective homomorphism with $\delta(\mathbb{Q})$ as the kernel. Thus $\widehat{\mathbb{Q}}$ is isomorphic to $\frac{\mathbb{Q}_{\mathbb{A}}}{\delta(\mathbb{Q})}$.
It is also stated in the article that if K is any finite field extension of $\mathbb{Q}$, then $\widehat{\mathbb{K}}$ is isomorphic to $\frac{\mathbb{K}_{\mathbb{A}}}{\delta(\mathbb{K})}$.

K. Conrad "The character group of Q." Unpublished (2010)

Solenoid in terms of dual groups

Theorem (Sharan, Raja, 2017)

Let Σ, n_p and D_∞ be defined as above.
Then $\Sigma = \frac{\mathbb{Q}_\mathbb{A}}{\delta(\mathbb{Q})+L}$, where $L = \prod_{p\leq\infty} U_p$ and

$$U_p = \begin{cases} (0) & \text{if } p \in D_\infty \cup \{\infty\} \\ p^{n_p}\mathbb{Z}_p & \text{if } p \notin D_\infty \cup \{\infty\} \end{cases}.$$

Theorem (Sharan, Raja, 2017)

Let Σ, L and D_∞ be defined as in above.

$$P(\alpha) = \frac{\delta(\mathbb{Q})+\prod' \mathbb{Q}_p}{\delta(\mathbb{Q})+L}$$

, where $\prod' \mathbb{Q}_p := \{x \in \mathbb{Q}_\mathbb{A} : x_p = 0$ for every $p \in D_\infty \cup \{\infty\}$ and $x_p \in p^{n_p}\mathbb{Z}_p$ for all but finitely many p in $P \setminus D_\infty\}$.

The above characterizations depend upon the description of the subgroups of $\mathbb{Q}$ using the notion of $p-$heights. However, no such description is available for the subgroups of $\mathbb{Q}^n$ for $n > 1$.
In fact, *[Kechris]* [1] says that there is probably "no reasonably simple classification" of these groups.

[1] A. S. Kechris, On the classification problem for rank 2 torsion-free abelian groups, J. London Math. Soc. (2) **62** (2000), 437-450.

Solenoid as an inverse limit

Theorem (Sharan, Faiz, 2021)

Let ϕ be an automorphism of a one dimensional solenoid Σ_A induced by $\frac{\alpha}{\beta}$, where $A = (\beta b_k)$, each b_k being co-prime to β. For each $l \in \mathbb{N}$, define $U_l = \bigcap_{p \in P} \left(\frac{1}{p^{e_{p,l}}} \mathbb{Z}_p \cap \mathbb{Q} \cap S^1 \right)$, where $p^{e_{p,l}} = \frac{1}{|\alpha^l - \beta^l|_p}$. If $\gamma_{k,l} : U_l \to U_l$ is the map defined as $\gamma_{k,l}(x) = \beta b_k x (mod\ 1)$ for each $k \in \mathbb{N}$ and $l \in \mathbb{N}$, then

$$P(\phi) = \bigcup_{l=1}^{\infty} \lim_{\substack{\leftarrow \\ k}} (U_l, \gamma_{k,l}).$$

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$$P(\phi) = \bigcup_{l=1}^{\infty} \lim_{\substack{\leftarrow \\ k}} (U_l, \gamma_{k,l}).$$

Remark

The set of periodic points of period l is equal to $\lim\limits_{\substack{\leftarrow \\ k}} (U_l, \gamma_{k,l})$.

Here U_l is a subgroup of S^1 and the map $\gamma_{k,l}$ is the restriction of γ_k to U_l, where γ_k is a map on S^1 such that $\Sigma_{(nb_k)} = \lim\limits_{\substack{\leftarrow \\ k}} (S^1, \gamma_k)$.

Solenoid as an inverse limit

Theorem (Sharan, Faiz, 2021)

Let ϕ be an automorphism of a one dimensional solenoid Σ_A induced by $\frac{\alpha}{\beta}$ and for every $l \in \mathbb{N}$, let $e_{p,l} = \frac{1}{|\alpha^l - \beta^l|_p}$. Then the number of periodic points of ϕ with a period l is $\prod_{p \notin D_\infty^{(S)}} p^{e_{p,l}}$.

Remark

The above theorem about the number of periodic points, which follows from the above description, is in accordance with a similar result in [Richard Miles, "Periodic points of endomorphisms on solenoids and related groups" *Bulletin of the London Mathematical Society.* 2008, 40(4): 696-704.]

Higher dimensional Solenoids

We now extend our result about periodic points to some automorphisms of certain higher dimensional solenoids i.e n-dimensional solenoids which are conjugate to product of "n" one-dimensional solenoids.

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n-dimensional solenoids

For a positive integer $n > 1$, let $\pi^n : \mathbb{R}^n \to \mathbb{T}^n$ be the homomorphism defined as

$\pi^n((x_1, x_2, ..., x_n)) = (x_1 \ (mod\ 1), x_2 \ (mod\ 1), ..., x_n \ (mod\ 1))$.

Let $\overline{M} = (M_k)_{k=1}^{\infty} = (M_1, M_2, ...)$ be a sequence of $n \times n$ matrices with integer entries and non-zero determinant.

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Let $\overline{M} = (M_k)_{k=1}^{\infty} = (M_1, M_2, ...)$ be a sequence of $n \times n$ matrices with integer entries and non-zero determinant. Then, the $n-$dimensional solenoid $\sum_{\overline{M}}$ is defined as
$\sum_{\overline{M}} = \{(\mathbf{x}_k) \in (\mathbb{T}^n)^{\mathbb{N}_0} : \pi^n(M_k \mathbf{x}_k) = \mathbf{x}_{k-1} \text{ for every } k \in \mathbb{N}\}$.
In other words, $\sum_{\overline{M}} = \varprojlim_{k} (\mathbb{T}^n, \delta_k)$, where $\delta_k : \mathbb{T}^n \to \mathbb{T}^n$ is defined as $\delta_k(\mathbf{x}) = \pi^n(M_k \mathbf{x})$

Higher dimensional Solenoids

Theorem (Sharan, Faiz, 2021)

For each $l \in \mathbb{N}$, *define* $V_l = \prod_{i=1}^{n} \left(\bigcap_{p \in P} \left(\frac{1}{p^{e_{p,l,i}}} \mathbb{Z}_p \cap \mathbb{Q} \cap S^1 \right) \right)$, *where* $p^{e_{p,l,i}} = \frac{1}{|\alpha_i^l - \beta_i^l|_p}$. *If* $\delta_{k,l} : V_l \to V_l$ *is the map defined as* $\delta_{k,l}(\mathbf{x}) = \pi^n(M_k \mathbf{x})$ *for each* $k \in \mathbb{N}$ *and* $l \in \mathbb{N}$, *then*

$$P(\phi) = \bigcup_{l=1}^{\infty} \lim_{\substack{\leftarrow \\ k}} (V_l, \delta_{k,l}).$$

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$$P(\phi) = \bigcup_{l=1}^{\infty} \varprojlim_{k} (V_l, \delta_{k,l}).$$

Remark

The set of periodic points of ϕ with a period l is equal to $\varprojlim_{k} (V_l, \delta_{k,l})$. Here, V_l is a subgroup of $\mathbb{T}^n$ and $\delta_{k,l}$ is the restriction of δ_k to V_l, where each δ_k is a map on $\mathbb{T}^n$ such that $\sum_{\overline{M}} = \varprojlim_{k} (\mathbb{T}^n, \delta_k)$.

Previous Work on Periodicity of Solenoidal Automorphisms

- In paper [5], the authors give a characterization of the sets of periodic points of automorphisms on the following solenoids.
  - n-dim Tori ($\widehat{\Sigma} = \mathbb{Z}^n$)
  - n-dim Full Solenoids ($\widehat{\Sigma} = \mathbb{Q}^n$)
  - 1-dim Solenoids ($\mathbb{Z} \leq \widehat{\Sigma} \leq \mathbb{Q}$)
- In the paper [4], the authors give a characterization of the sets of periodic points of automorphisms on the following solenoids using the concept of inverse limits.
  - 1-dim Solenoids
  - n-dim Solenoids which are product of "n" : 1-dim Solenoids

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[4] S. Gopal and F. Imam, *Periodic points of solenoidal automorphisms in terms of inverse limits*, Applied General Topology **22(2)** (2021), 321-330.

[5] S. Gopal and C. R. E. Raja, *Periodic points of solenoidal automorphisms*, Topology Proceedings **50** (2017), 49 - 57.

Preliminaries

- A finite algebraic extension of the field of rational numbers $\mathbb{Q}$ is defined as an algebraic number field $\mathbb{K}$.
- We denote by $P^{\mathbb{K}}$, the set of all places of $\mathbb{K}$, i.e, the equivalence classes of valuations of $\mathbb{K}$ (where two valuations ϕ_1 and ϕ_2 are said to be equivalent if there is an $s > 0$ such that $\phi_1(r) = \phi_2(r)^s$ for every $r \in \mathbb{K}$). A place is called finite if it contains a non-archimedian valuation and infinite otherwise.
- The collection of finite places will be denoted by $P_f^{\mathbb{K}}$ whereas $P_\infty^{\mathbb{K}}$ denotes the set of infinite places. It may be noted that $P_\infty^{\mathbb{K}}$ is a finite set.
- For each $v \in P^{\mathbb{K}}$, $\mathbb{K}_v$ denotes the completion of $\mathbb{K}$ with respect to v and $\mathfrak{R}_v = \{x \in \mathbb{K}_v : |x|_v \leq 1\}$. $\mathfrak{R}_v$ is always a compact subset of $\mathbb{K}_v$ and when $v \in P_f^{\mathbb{K}}$, $\mathfrak{R}_v$ is an open, unique maximal compact subring of $\mathbb{K}_v$. We also consider $\mathfrak{R}_v^* := \{x \in \mathfrak{R}_v : |x|_v = 1\}$ in our discussion.
- The adele ring of $\mathbb{K}$, denoted by $\mathbb{K}_{\mathbb{A}}$ is then defined as $\mathbb{K}_{\mathbb{A}} = \left\{(x_v) \in \prod_{v \in \mathbb{P}^{\mathbb{K}}} \mathbb{K}_v / \ x_v \in \mathfrak{R}_v \text{ for all but finitely many } v \in P_f^{\mathbb{K}} \right\}$.

Preliminaries

- If $\mathbb{K}$ is an algebraic number field, then for each $p \in P^{\mathbb{Q}}$, there exists finitely many $v \in P^{\mathbb{K}}$ such that v lies above p (denoted as $v|_p$).
- We consider solenoids of any arbitrary dimension n such that $\widehat{\Sigma}$ is an additive subgroup of an algebraic number field $\mathbb{K}$.
- Now consider $\mathbb{K}_{\mathbb{A}}$, the ring of adeles of $\mathbb{K}$. For any $p \in P^{\mathbb{Q}}, \mathbb{Z}_p$ can be considered as a subring of $\mathbb{Q}_{\mathbb{A}}$ by identifying $c \in \mathbb{Z}_p$ with $x \in \mathbb{Q}_{\mathbb{A}}$, when $x_p = c$ and $x_q = 0$ for $q \neq p$.
- Similarly $\prod_{v|_p} \mathbb{K}_v$ can be considered as a subring of $\mathbb{K}_{\mathbb{A}}$ by identifying $\prod_{v|_p} a_v \in \prod_{v|_p} \mathbb{K}_v$ with $b \in \mathbb{K}_{\mathbb{A}}$, when $b_v = a_v$ for $v|_p$ and $b_w = 0$ otherwise.
- From Lemma 6.101 of [Kato], it follows that there is an isomorphism (of topological groups) $\alpha\ : \mathbb{K}_{\mathbb{A}} \ \to (\mathbb{Q}_{\mathbb{A}})^n$ such that $\alpha\left(\prod_{v|_p} \Re_v\right)$ is equal to $(\mathbb{Z}_p)^n$ for almost all finite p.

Notations and Assumptions

- We further assume that $\alpha\left(\prod_{v|_p} \mathfrak{R}_v\right) = (\mathbb{Z}_p)^n$ for all the finite places. We write $\alpha(x) = \left(x^{(1)}, x^{(2)}, \cdots, x^{(n)}\right) \in (\mathbb{Q}_{\mathbb{A}})^n$, for each $x \in \mathbb{K}_{\mathbb{A}}$ and write $x^{(j)} = \left(x_p^{(j)}\right)_{p\in P^{\mathbb{Q}}}$, for each $x^{(j)} \in \mathbb{Q}_{\mathbb{A}}$.
- For every $r \in \mathbb{K}$, we write $\beta(r) = \left(r^{(1)}, r^{(2)}, \cdots, r^{(n)}\right) \in \mathbb{Q}^n$ where $r = \sum_{i=1}^{n} r^{(i)}\alpha_i$ and $\{\alpha_1, \alpha_2, \cdots, \alpha_n\}$ is a $\mathbb{Q}$-basis for $\mathbb{K}$. Then, β is an isomorphism from $\mathbb{K}$ to $\mathbb{Q}^n$. We further assume that $\beta(\widehat{\Sigma})$ is a $\mathbb{Z}^n$-module and also $\mathbb{Z}^n \subseteq \beta(\widehat{\Sigma})$.
- For $a = (a_v)_{v\in P^{\mathbb{K}}} \in \mathbb{K}_{\mathbb{A}}$, let $\overline{a}_p = \prod_{v|_p} a_v \in \prod_{v|_p} \mathbb{K}_v$, for every $p \in P^{\mathbb{Q}}$. We know that $\prod_{v|_p} \mathbb{K}_v$ is a vector space over $\mathbb{Q}_p$. It follows from Lemma 6.69 and 6.101 of [Kato] that the $\mathbb{Q}_p$-coordinates of $\overline{a}_p$ are same as $\left(a_p^{(1)}, a_p^{(2)}, \cdots, a_p^{(n)}\right)$, where $\left(a^{(1)}, a^{(2)}, \cdots, a^{(n)}\right) = \alpha(a)$ and $a^{(j)} = \left(a_q^{(j)}\right)_{q\in P^{\mathbb{Q}}}$.

Definitions

- Consider the map $\eta : \mathbb{Q}_{\mathbb{A}} \to \widehat{\mathbb{Q}}$ given by $\eta(x) = \eta_x$, where $\eta_x : \mathbb{Q} \to S^1$ is defined as $\eta_x(r) = e^{-2\pi i x_\infty r} \cdot \prod_{p<\infty} e^{2\pi i \{x_p r\}_p}$ and $x = (x_p)_{p \in P^{\mathbb{Q}}}$. It is known that this map η is a surjective homomorphism.
- Now, consider the map $\xi : (\mathbb{Q}_{\mathbb{A}})^n \to \widehat{\mathbb{Q}^n}$ given by $\xi(\bar{x}) = \xi_{\bar{x}}$, where $\xi_{\bar{x}} : \mathbb{Q}^n \to S^1$ is defined as $\xi_{\bar{x}}(\bar{r}) = \eta_{x^{(1)}}(r^{(1)}) \cdot \eta_{x^{(2)}}(r^{(2)}) \cdots \eta_{x^{(n)}}(r^{(n)})$, where $\bar{x} = (x^{(1)}, x^{(2)}, \cdots, x^{(n)}) \in (\mathbb{Q}_{\mathbb{A}})^n$ and $\bar{r} = (r^{(1)}, r^{(2)}, \cdots, r^{(n)}) \in \mathbb{Q}^n$. Observe that ξ is a homomorphism.
- Note that $\xi_{(\bar{x})}(\bar{r}) = e^{-2\pi i \sum_{j=1}^{n} x_\infty^{(j)} r^{(j)}} \cdot \prod_{p<\infty} e^{2\pi i \sum_{j=1}^{n} \{x_p^{(j)} r^{(j)}\}_p}$.
- Now, define $\omega : \mathbb{K}_{\mathbb{A}} \to \widehat{\mathbb{Q}^n}$ as $\omega(a) = \omega_a$, where $\omega_a = \xi \circ \alpha(a)$; in other words, if $a \in \mathbb{K}_{\mathbb{A}}$ and $\alpha(a) = (a^{(1)}, a^{(2)}, \cdots, a^{(n)})$, then $\omega_a(\bar{r}) = e^{-2\pi i \sum_{j=1}^{n} a_\infty^{(j)} r^{(j)}} \cdot \prod_{p<\infty} e^{2\pi i \sum_{j=1}^{n} \{a_p^{(j)} r^{(j)}\}_p}$.

Definitions

- Since ξ and α are homomorphisms, ω is also a homomorphism. Finally, define $\psi : \mathbb{K}_{\mathbb{A}} \to \widehat{\mathbb{K}}$ as $\psi(a) = \psi_a,$ for every $a \in \mathbb{K}_{\mathbb{A}}$, where $\psi_a : \mathbb{K} \to S^1$ is given by $\psi_a(r) = \omega_a \circ \beta(r)$, for every $r \in \mathbb{K}$.
- Note that if $\alpha(a) = (a^{(1)}, a^{(2)}, \cdots, a^{(n)}) \in (\mathbb{Q}_{\mathbb{A}})^n$ and $\beta(r) = (r^{(1)}, r^{(2)}, \cdots, r^{(n)}) \in \mathbb{Q}^n$ then $\psi_a(r) = w_a \circ \beta(r) = w_a(r^{(1)}, r^{(2)}, \ldots, r^{(n)}) = \xi_{\alpha(a)}(r^{(1)}, r^{(2)}, \ldots, r^{(n)}) = \xi_{(a^{(1)}, a^{(2)}, \ldots, a^{(n)})}(r^{(1)}, r^{(2)}, \ldots, r^{(n)}) =$
$$e^{-2\pi i \sum\limits_{j=1}^{n} a_\infty^{(j)} r^{(j)}} \cdot \prod_{p<\infty} e^{2\pi i \sum\limits_{j=1}^{n} \{a_p^{(j)} r^{(j)}\}_p}.$$
- Note that ψ is a homomorphism.

Proposition

ψ is a surjective homomorphism that is trivial on $i(\mathbb{K})$.

- Since $\widehat{\Sigma}$ is a subgroup of $\mathbb{K}$, we have $\widehat{\widehat{\Sigma}} = \widehat{\mathbb{K}}/ann(\widehat{\Sigma})$ and thus, $\Sigma = \widehat{\mathbb{K}}/ann(\widehat{\Sigma})$. Define $\psi' : \mathbb{K}_{\mathbb{A}} \to \Sigma$ as $\psi' = \pi \circ \psi$, where $\pi : \widehat{\mathbb{K}} \to \Sigma$ is the quotient map.
- Since π and ψ are surjective, ψ' is surjective. We will now find Ker $\psi^{'}$ and thus obtain Σ as a quotient of $\mathbb{K}_{\mathbb{A}}$.
- For every $p \in P_f^{\mathbb{Q}}$ and $1 \leq j \leq n$, define $m_p^{(j)} = \sup\{|r^{(j)}|_p \ : \ r \in \widehat{\Sigma}\}$, where $\beta(r) = \left(r^{(1)}, r^{(2)}, \cdots, r^{(n)}\right)$.
- Since $\mathbb{Z}^n \subset \beta(\widehat{\Sigma})$, we have $r = \beta^{-1}(0, \cdots, p, \cdots, 0) \in \widehat{\Sigma}$ and thus $|r^{(j)}|_p = |p|_p = \frac{1}{p} \neq 0$ concluding that $m_p^{(j)} \neq 0$.

- Let $n_p^{(j)} = \begin{cases} \frac{1}{m_p^{(j)}} & \text{if } m_p^{(j)} < \infty \\ 0 & \text{if } m_p^{(j)} = \infty \end{cases}$ and
  $D = \{p \in P_f^{\mathbb{Q}} : m_p^{(j)} = \infty \text{ for every } 1 \leq j \leq n\}$.
- Now, define a subgroup U_p of $\prod_{v|_p} \mathbb{K}_v$ for every $p \in P^{\mathbb{Q}}$ as
  $U_p =$
  $\begin{cases} (0) & \text{for } p \in D \cup \{\infty\} \\ \{x \in \prod_{v|_p} \mathbb{K}_v : |x^{(j)}|_p \leq n_p^{(j)} \text{ for every } j\} & \text{for } p \notin D \cup \{\infty\} \end{cases}$,
  where $x^{(1)}, x^{(2)}, \cdots, x^{(n)}$ are $\mathbb{Q}_p-$coordinates of x.
- Finally, define $V = i(\mathbb{K}) + \prod_{p \in P^{\mathbb{Q}}} U_p$.

Results

Theorem

Σ *is isomorphic to* $\mathbb{K}_{\mathbb{A}}/V$.

- We now describe the periodic points of some automorphisms of Σ. Fix an element $(d) = (d^{(1)}, d^{(2)}, \cdots, d^{(n)}) \in \mathbb{Q}^n$ such that for every j, $|d^{(j)}| \neq 0$ and $|d^{(j)}|_p = 1$ for $p \notin D \cup \{\infty\}$.
- Define a map $M_d : \mathbb{K}_{\mathbb{A}} \to \mathbb{K}_{\mathbb{A}}$ as $\alpha^{-1} \circ m_d \circ \alpha$, where $m_d : (\mathbb{Q}_{\mathbb{A}})^n \to (\mathbb{Q}_{\mathbb{A}})^n$ is given by $m_d(a^{(1)}, a^{(2)}, \cdots, a^{(n)}) = ((d^{(1)} a_p^{(1)})_p, (d^{(2)} a_p^{(2)})_p, \cdots, (d^{(n)} a_p^{(n)})_p)$.
- Note that $(d^{(j)} a^{(j)} p)_p = (d^{(j)} a_\infty^{(j)}, d^{(j)} a_2^{(j)}, d^{(j)} a_3^{(j)}, \cdots)$.
- m_d is an isomorphism and thus $M_d = \alpha^{-1} \circ m_d \circ \alpha$ is an automorphism of $\mathbb{K}_{\mathbb{A}}$.

Results

Proposition

$M_d(V) = V = i(\mathbb{K}) + \prod U_p.$

M_d is an automorphism on $\mathbb{K}_{\mathbb{A}}$ and V is an M_d-invariant subgroup of $\mathbb{K}_{\mathbb{A}}$, M_d induces an automorphism of Σ, say $\overline{M_d}$.

Theorem (Description of Periodic Points)

The set of periodic points of $\overline{M_d}$, where $d^{(j)} \neq \pm 1$ for every $1 \leq j \leq n$, is given by $P(\overline{M_d}) = \frac{i(\mathbb{K})+\prod' \mathbb{K}_v}{V}$, where $\prod' \mathbb{K}_v = \left\{ x \in \mathbb{K}_{\mathbb{A}} : \text{for every } 1 \leq j \leq n, x_p^{(j)} = 0 \text{ whenever } p \in D \cup \{\infty\} \text{ and } |x_p^{(j)}|_p \leq n_p^{(j)} \text{ for all but finitely many } p \notin D \cup \{\infty\} \right\}$.

SHUKRIYA!

(THANK YOU)