Topologies related to (I)-envelopes

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(I)-generation and (I)-envelopes

Theorem [Fonf & Lindenstrauss, 2003]

Let X be a (real) Banach space, $K \subset X^*$ convex, weak*-compact, $B \subset K$ a boundary, i.e.,

$$\forall x \in X \, \exists b \in B \colon b(x) = \max \langle K, x \rangle ,$$

then B (I)-generates K, i.e.,

$$B = \bigcup_n B_n \Rightarrow K = \overline{\operatorname{conv} \bigcup_n \overline{\operatorname{conv} B_n}^{w^*}}^{\|\cdot\|} .$$

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Definition [OK, 2007]

Let X be a Banach space and $A \subset X^*$. The (I)-envelope of A is defined by

$$\text{(I)-env}(A) = \bigcap \left\{ \overline{\operatorname{conv} \bigcup_{n=1}^{\infty} \overline{\operatorname{conv} A_n}^{w^*}}^{\|\cdot\|} \; ; \; A = \bigcup_{n=1}^{\infty} A_n \right\} .$$

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- (I)-env$(A) =$
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- (I)-env((I)-env$(A)) =$ (I)-env(A).

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- $\text{(I)-env}(\text{(I)-env}(A)) = \text{(I)-env}(A)$.
- A is norm-separable $\Rightarrow \text{(I)-env}(A) = \overline{\operatorname{conv} A}^{\|\cdot\|}$.

Some applications of (I)-envelopes

- A proof of James' compactness theorem in separable Banach spaces and in some more general Banach spaces (with weak*-sequentially compact dual unit ball).

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- A characterization of Grothendieck spaces:

$$X \text{ is Grothendieck} \Leftrightarrow \text{(I)-env}(X) = X^{**}.$$

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- A geometric reformulation of Simons equality.
- A characterization of Grothendieck spaces:
  $$X \text{ is Grothendieck} \Leftrightarrow (\mathrm{I})\text{-env}(X) = X^{**}.$$
- [Bendová 2014] A characterization of quantitatively Grothedieck spaces:
  $$X \text{ is } c\text{-Grothendieck} \Leftrightarrow (\mathrm{I})\text{-env}(B_X) \supset \frac{1}{c} \cdot B_{X^{**}}.$$

Our main problem

Question

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Sometimes yes.

- X^* separable $\Rightarrow \tau = \|\cdot\|$ works.

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Remark

Sometimes yes.

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- [Fonf & Lindenstrauss, 2003]
  X separable, $\ell_1 \not\subset X \Rightarrow \tau = \|\cdot\|$ works.

Separation lemma

Lemma [OK, 2007]

Let X be a Banach space, $A \subset X^*$ and $\eta \in X^*$. The following assertions are equivalent.

1. $\eta \notin$ (I)-env(A).
2. There is a sequence (x_n) in B_X such that
$$\sup_{\xi\in A} \limsup_{n\to\infty} \operatorname{Re}\xi(x_n) < \inf_{n\in\mathbb{N}} \operatorname{Re}\eta(x_n).$$
3. There is a sequence (x_n) in B_X such that
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Locally convex case – a positive result

Notation

For a Banach space X set

$$B_1(X) = \{x^{**} \in X^{**} \; ; \; \exists (x_n) \text{ a sequence in } X : x_n \xrightarrow{w^*} x^{**}\},$$

$$C(X) = \{x^{**} \in X^{**} \; ; \; \exists C \subset X \text{ countable} : x^{**} \in \overline{C}^{w^*}\}.$$

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Proposition

Let X be a Banach space and $A \subset X^*$. Then

$$\overline{\operatorname{conv} A}^{\sigma(X^*, C(X))} \subset (\mathrm{I})\text{-env}(A) \subset \overline{\operatorname{conv} A}^{\sigma(X^*, B_1(X))}.$$

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Let X be a Banach space and $A \subset X^*$. Then

$$\overline{\operatorname{conv} A}^{\sigma(X^*,C(X))} \subset \text{(I)-env}(A) \subset \overline{\operatorname{conv} A}^{\sigma(X^*,B_1(X))}.$$

Theorem

Assume that X is a Banach space not containing an isomorphic copy of ℓ^1. Then $B_1(X) = C(X)$, hence for any set $A \subset X^*$ we have

$$\text{(I)-env}\, A = \overline{\operatorname{conv} A}^{\sigma(X^*,B_1(X))}.$$

Locally convex case – a necessary condition

Lemma

Let X be a Banach space.

- Let $Y \subset X^*$ be a linear subspace. Then (I)-env$(Y) = \{\eta \in X^* \; ; \; \forall (x_n)$ sequence in B_X :

$$x_n \xrightarrow{\sigma(X,Y)} 0 \Rightarrow \eta(x_n) \to 0\}.$$

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$$x_n \xrightarrow{\sigma(X,Y)} 0 \Rightarrow \eta(x_n) \to 0\}.$$
- Assume that $x^{**} \in X^{**}$. Then
$$\text{(I)-env}(\ker x^{**}) = \begin{cases} \ker x^{**} & x^{**} \in B_1(X) \\ X^* & x^{**} \notin B_1(X). \end{cases}$$

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Corollary

Let X be a Banach space. Assume that there is a locally convex topology τ on X^* such that $\text{(I)-env}(A) = \overline{\text{conv}\, A}^{\tau}$ for each $A \subset X^*$. Then $(X^*, \tau)^* = B_1(X)$.

Locally convex case – a counterexample

Proposition

Let X be a Banach space. The following assertions are equivalent.

1. There is a locally convex topology τ on X^* such that (I)-env$(A) = \overline{\text{conv}\, A}^{\tau}$ for each $A \subset X^*$.
2. (I)-env$(A) = \overline{\text{conv}\, A}^{\sigma(X^*, B_1(X))}$ for each $A \subset X^*$.

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Example

There is no locally convex topology τ on $(\ell^1)^*$ such that (I)-env$(A) = \overline{\text{conv}\, A}^{\tau}$ for each $A \subset (\ell^1)^*$.

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Proof

- $B_1(\ell^1) = \ell^1 \Rightarrow \sigma((\ell^1)^*, B_1(\ell^1)) = w^*$

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Proof

- $B_1(\ell^1) = \ell^1 \Rightarrow \sigma((\ell^1)^*, B_1(\ell^1)) = w^*$
- $A = c_0 \subset \ell^\infty = (\ell^1)^*$

Some more counterexamples

Similar examples may be found in X^* in the following cases:

- X is a non-reflexive weakly sequentially complete Banach space;

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Similar examples may be found in X^* in the following cases:

- X is a non-reflexive weakly sequentially complete Banach space;
- X contains a complemented copy of ℓ^1;
- $X = C(K)$ where K is an uncountable metrizable compact space.

Locally convex case – open problems

Question

Is it true that $\ell_1 \not\subset X$ if and only if there is a locally convex topology τ on X^* such that $(\mathrm{I})\text{-env}(A) = \overline{\operatorname{conv} A}^{\tau}$ for each $A \subset X^*$?

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Question

Is there a locally convex topology on X^* such that $(I)\text{-env}(A) = \overline{\text{conv}\, A}^{\tau}$ for each bounded $A \subset X^*$?

Topological case – a naive try

Definition

For $A \subset X^*$ define

- (I)-cl$(A) = \bigcap \left\{ \overline{\bigcup_{n=1}^{\infty} \overline{A_n}^{w^*}}^{\|\cdot\|} \; ; \; A = \bigcup_{n=1}^{\infty} A_n \right\}$

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Facts

- (I)-cl and (I)-ccl are idempotent closure operators.

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- $\text{(I)-cl}(A) = \text{(I)-ccl}(A)$.

Example

Let $X = C[0,1]$ and let A consist of countably supported probabilities on $[0,1]$. Then A is convex and
$\text{(I)-cl}(A) = \text{(I)-ccl}(A) = A \subsetneq P[0,1] = \text{(I)-env}(A)$.

Topological case – the maximal closure operator

Definition

For $A \subset X^*$ define

$$\mathrm{cl}_{IF}(A) = \bigcap \left\{ \bigcup_{k=1}^{n} \text{(I)-env}(A_k)\ ;\ A = \bigcup_{k=1}^{n} A_k \right\}$$

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- cl_{IF} is a closure operator.

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Topological case – the key problem

Theorem

Let X be a Banach space. TFAE:

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3. $\text{(I)-env}(A_1 \cup \cdots \cup A_n) = \text{(I)-env}(A_1) \cup \cdots \cup \text{(I)-env}(A_n)$ whenever $A_1, \ldots, A_n \subset X^*$ are convex and $A_1 \cup \cdots \cup A_n$ is convex as well.

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Does 3 hold for any Banach space? Does it hold for $X = \ell^1$?

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Thank you for your attention.