ON μ-COMPLETENESS OF UNIFORM SPACES AND UNIFORMLY CONTINUOUS MAPPINGS

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Abstract

In this talk μ-complete uniform spaces are studied, i.e. those uniform spaces, where every Cauchy filters with base of cardinality $\leq \mu$ are converging in it. In particular some new concepts are introduced such as index $ic_{\mu}(X, U)$ of μ-completeness of the uniform space (X, U) and Dieudonne μ-complete space of a Tychonoff space X, and also index $ic_{\mu}(f)$ of μ-completeness of the uniformly continuous mapping $f : (X, U) \to (Y, V)$ between uniform spaces (X, U) and (Y, V), and some properties of these concepts are established. Namely

1. $ic_{\mu}(X, U) = 1$ iff (X, U) is uniformly locally μ-compact space;
2. $ic_{\mu}(f) = 1$ iff f is uniformly locally μ-quasi-perfect mapping;
3. Tychonoff space (X, U) is Dieudonne μ-complete space iff the uniform space(X, U_X) with the universal uniformity U_X is μ-complete.

Introduction

Throughout the talk "a uniformity" means a uniformity U defined by using covers. For a uniformity Uon a set X, τ_U denotes the topology on X induced by U. For a Tychonoff space $X$by U_X we denote the universal uniformity of X.

A Tychonoff space X is called:

1. *μ-compact*, if every its open cover of cardinality $\leq \mu$ has a finite open refinement, [1];
2. *Dieudonne complete space*, if there is a uniformity U agreed with the topology of a space X such that a uniform space (X, U) is complete [5].

A uniform space (X, U) is called:

1. *uniformly locally compact*, if there exists a uniform covering consisting of compact subsets [5];
2. *μ-complete*, if every Cauchy filter with base of a cardinality $\leq \mu$ converges in it [3].

Definitions and denotations

Let $f:(X,U)\to(Y,V)$ be a uniformly continuous mapping between uniform spaces (X,U) and (Y,V). The pseudouniformity $U_f\subset U$ is called a *base* of the uniformly continuous mapping f, if for any $\alpha\in U$ there are $\beta\in V$ and $\gamma\in U_f$ such that $f^{-1}\beta\wedge\gamma\succ\alpha$ [2].

The mapping f is called:

1. *precompact*, if it has precompact base $U_f\subset U$, [2];
2. *uniformly perfect*, if it is both precompact and perfect, [2];
3. *uniformly open*, if f maps each open cover $\alpha\in U$ to an open cover $f\alpha\in V$, [3];
4. *complete*, if for every Cauchy filter F of a uniform space (X,U) for it the image fF converges in (Y,V), converges in (X,U) [3].

For covers $\alpha$and β of a set X, the symbol $\alpha\succ\beta$ means that the cover α is a refinement of the cover β, i.e. for any $A\in\alpha$ there exists $B\in\beta$ such that $A\subset B$. The cover consisting of all sets of form $\{A\bigcap B: A\in\alpha, B\in\beta\}$ is said to be the inner intersection of the covers α and β, and it is denoted by $\alpha\wedge\beta$.

Index of μ-completeness of uniform spaces. Main Results.

In uniform topology, one of the central concepts is the concept of completeness of uniform spaces. There are various kinds of the completeness of uniform spaces such as: completeness, weak completeness, τ-completeness, μ-completeness and others.
Below the index of μ-completeness of uniform spaces are studied.
μ-completeness of uniform spaces was introduced by A. Borubaev [2].
The following theorem characterizes μ-compact spaces by means of uniform structures.

Theorem 2.1

A Tychonoff space X is μ-compact if and only if for each uniformity U the uniform space (X, U) is μ-complete.

Every complete space is μ-complete, however, the following theorem shows that in the class of uniform spaces with a weight $w(U) \leq \mu$ the concepts of completeness and μ-completeness coincide.

Main Results

Theorem 2.2

If uniform space (X,U) is μ-complete and $w(U) \leq \mu$, then the space (X,U) is complete.

Theorem 2.3

Let $f:(X,U) \to (Y,V)$ be a perfect uniformly continuous mapping of a uniform space (X,U) onto μ-complete space (Y,V). Then the uniform space (X,U) is also μ-complete and $ic_\mu(U) \leq ic_\mu(V)$.

Corollary 2.1.

Let $f:(X,U) \to (Y,V)$ be a uniformly perfect mapping of a uniform space (X,U) onto μ-complete space (Y,V). Then the uniform space (X,U) is also μ-complete and $ic_\mu(U) \leq ic_\mu(V)$.

Definition 2.1.

A Tychonoff space X is called *Dieudonne μ-complete* space, if there is a uniformity U agreed with the topology of a space X such that a uniform space (X, U) is μ-complete.

It is clear that every Dieudonne complete space is Dieudonne μ-complete.

Theorem 2.4.

Tychonoff space X is Dieudonne μ-complete iff a uniform space (X, U_X) with the universal uniformity U_X is μ-complete.

Let (X, U) be uniform space and $H \subset U$ the system of uniform covers. The filter F is called H-Cauchy filter if $\alpha \bigcap F \neq \emptyset$ for any $\alpha \in H$ [2].

Definition 2.2.

A uniform space (X, U) is called *$H-\mu$-complete*, if every H-Cauchy filter F with a base of a cardinality $\leq \mu$ has at least one condensation point.

If the system $H \subset U$ is the base for the uniformity U, then Cauchy H-filter is the usual Cauchy filter in (X, U) and every condensation point of Cauchy filter is its limit point, i.e. $H - \mu$-completeness of a uniform space (X, U) is transformed into its usual μ-completeness.

Definition 2.3.

The smallest cardinal number η is called index of μ-completeness of the uniform space (X, U), if there exists a system $H \subset U$ such that $|H| = \eta$ and (X, U) is $H - \mu$-complete. The index of μ-completeness of a uniform space (X, U) is denoted by $ic_{\mu}(U)$.

For each μ-complete uniform space (X, U) its index of μ-completeness $ic_{\mu}(U)$ is defined and either $ic_{\mu}(U) = 1$ or $\aleph_0 \leq ic_{\mu}(U) \leq w(U)$.

A uniform space (X, U) is called *uniformly locally μ-compact*, if there exists a uniform covering such that the closures of all its elements are μ-compact. Any uniformly locally compact uniform space (X, U) is uniformly locally μ-compact.

Theorem 2.5.

For a μ-complete uniform space (X, U) the following are equivalent:

1. $ic_{\mu}(U) = 1$;
2. The uniform space (X, U) is uniformly locally μ-compact.

Index of μ-completeness of uniformly continuous mappings

Last time many concepts and assertions of uniform topology are extended from spaces to uniformly continuous mappings. Here the uniform space is understood as the simplest uniformly continuous mapping of this uniform space into a one-point space.

The studies carried out revealed large uniform analogues of continuous mappings and made it possible to transfer to mappings many basic statements of the uniform topology of spaces, in the works of A.A. Borubaev, Z. Frolik, B.A. Pasynkov, A.S. Mishchenko, A.A. Chekeev and others. The method of transferring results from spaces to mappings is universal, and not simple, but it allows to generalize many results.

Below the index of μ-completeness is extended to mappings of uniform spaces.

A uniform continuous mapping $f : (X, U) \to (Y, V)$ between the uniform spaces (X, U) and (Y, V) is called μ-complete, if for every Cauchy filter F with base cardinality $\leq \mu$ of a uniform space (X, U) for it the image fF converges in (Y, V), converges in (X, U) [4].

Main Results

Any complete uniformly continuous mapping is μ-complete.
If $f : (X,U) \to (Y,V)$ is uniformly continuous mapping of μ-complete uniform space (X,U) into a uniform space (Y,V), then the mapping f is μ-complete. Conversely, if the mapping f of a uniform space (X,U) into one-point uniform space $Y = \{y\}$ is μ-complete, then the uniform space (X,U) is μ-complete.
Hence from the definition of the base of a uniformly continuous mapping that every uniformly continuous mapping possesses, generally speaking, many bases.
The least cardinal number τ that is the weight of some base U_f of the mapping f, is called the weight of the mapping $f$and it is denoted by $w(f)$.
If uniform continuous mapping $f : (X,U) \to (Y,V)$ is μ-complete and $w(f) \leq \mu$, then the map f is complete.

The following theorem shows that under μ-complete mappings, the μ-completeness of uniform spaces is preserved towards the inverse image.

Theorem 3.1.

If the mapping f and the space (Y, V) are μ-complete, then the space (X, U) is μ-complete too.

Definition 3.1.

A uniform continuous mapping $f : (X, U) \to (Y, V)$ between the uniform spaces (X, U) and (Y, V) is called *$H - \mu$-complete*, if for any H-Cauchy filter F with base of a cardinality $\leq \mu$ of a uniform space (X, U) for it the image fF converges in (Y, V), converges in (X, U).

Definition 3.2.

The smallest cardinal number η is called index of μ-completeness of a uniformly continuous mapping f, if there exists a system $H \subset U_f$ such that $|H| = \eta$ and f is $H - \mu$-complete. The index of μ-completeness of uniform continuous mapping f denoted by $ic_\mu(f)$.

Let $f:(X,U)\to(Y,V)$ be a continuous mapping between topological spaces (X,τ) and (Y,σ). A mapping f is called *μ-quasi-perfect*, if f is closed and a subset $f^{-1}y$ is μ-compact for any $y\in Y$.
$\aleph_0$-quasi-perfect mappings are called simply quasi-perfect mappings.
It is easy to establish that a mapping f is μ-quasi-perfect if and only if for any point $y\in Y$ and any covering λ of a cardinality $\leq\mu$ of a layer $f^{-1}y$ by open sets in X there exists a neighborhood V of the point y, the set $f^{-1}V$ over which is covered by a finite number of elements of λ.

Definition 3.3.

A uniformly continuous mapping $f:(X,U)\to(Y,V)$ between uniform spaces (X,U) and (Y,V) is said to be *uniformly locally μ-quasi-perfect*, if there exists a uniform covering $\alpha\in U$ such that a restriction $f|_A:(A,\tau_{U_A})\to(Y,\tau_V)$ is μ-quasi-perfect for any $A\in\alpha$.

Theorem 3.2.

For μ-complete uniformly continuous mapping $f : (X, U) \to (Y, V)$ between uniform spaces (X, U) and (Y, V) the following are equivalent:

1. $ic_\mu(f) = 1$;
2. the mapping f is uniformly locally μ-quasi-perfect.

Theorem 3.3.

Let $f : (X, U) \to (Y, V)$ be a uniformly open mapping and $ic_\mu(f) \leq \tau$. If $ic_\mu(V) \leq \tau$, then $ic_\mu(U) \leq \tau$.

References

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Thanks for attention.