Baire-one functions on topological spaces: some recent results and open questions

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- Baire-1 functions vs F_σ-measurable functions
- Homotopic Baire-1 class
- Fragmentability and extension property

BAIRE-ONE FUNCTIONS vs F_σ-MEASURABLE FUNCTIONS

Definitions and notations

- A function $f : X \to Y$ is *of the first Baire class*, $f \in \mathrm{B}_1(X, Y)$, if f is a pointwise limit of a sequence of continuous maps $f_n : X \to Y$
- A function $f : X \to Y$ is *F_σ-measurable*, $f \in \mathscr{F}_\sigma(X, Y)$, or *of the first Borel class*, if for any open subset V of Y there exists a sequence of closed sets in X such that $f^{-1}(V) = \bigcup_{n\in\omega} F_n$

$$B_1(\mathbb{R}, \mathbb{R}) = \mathscr{F}_\sigma(\mathbb{R}, \mathbb{R})$$

Lebesgue-Hausdorff Theorem

Theorem (Lebesgue, Hausdorff)

$$\mathscr{F}_\sigma(X,Y) = \mathrm{B}_1(X,Y)$$

- X is a metrizable space and $Y = [0,1]^\omega$, or
- X is a metrizable separable space with $\dim X = 0$ and Y is a metrizable separable space.

The case of connected Y

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Theorem (Fosgerau, Veselý, 1993)

For a Polish space Y the following conditions are equivalent:

1. Y is connected and locally path-connected,
2. $\mathscr{F}_\sigma(X, Y) = \mathrm{B}_1(X, Y)$ for any perfectly normal X,
3. $\mathscr{F}_\sigma([0,1], Y) = \mathrm{B}_1([0,1], Y)$.

The case of disconnected Y. Necessary condition

- $f : X \to Y$ is functionally F_σ-measurable, $f \in \mathscr{F}_\sigma^*(X, Y)$, if for any open set $V \subseteq Y$ there exists a sequence $(F_n)_{n\in\omega}$ of zero-sets in X such that $f^{-1}(V) = \bigcup_{n\in\omega} F_n$

The case of disconnected Y. Necessary condition

- $f : X \to Y$ is functionally F_σ-measurable, $f \in \mathscr{F}^*_\sigma(X, Y)$, if for any open set $V \subseteq Y$ there exists a sequence $(F_n)_{n\in\omega}$ of zero-sets in X such that $f^{-1}(V) = \bigcup_{n\in\omega} F_n$
- $F^*_\sigma(X, Y) \subseteq F_\sigma(X, Y)$ and $F_\sigma(X, Y) = \mathscr{F}^*_\sigma(X, Y)$ for any normal space X

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- $F^*_\sigma(X,Y) \subseteq F_\sigma(X,Y)$ and $F_\sigma(X,Y) = \mathscr{F}^*_\sigma(X,Y)$ for any normal space X

Let X be a topological space, Y is disconnected space such that $\mathscr{F}^*_\sigma(X,Y) \subseteq B_1(X,Y)$. Then every zero-set $F \subseteq X$ can be written as

$$F = \bigcup_{k\in\omega} \bigcap_{n\in\omega} U_{kn},$$

where (U_{kn}) is a clopen set in X for all $k, n \in \omega$.

Almost strongly zero-dimensional space

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strongly zero-dim $\Rightarrow$ zero-dim $\Rightarrow$ AZD $\Rightarrow$ totally disconnected

Almost strongly zero-dimensional space

- We say that a set A is *a C_σ-set* if

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Definition

A completely regular space X is called *almost strongly zero-dimensional* (ASZD) if every zero-set $F \subseteq X$ is C_σ.

Y is metrizable and separable

Theorem (K., 2017)

If X is a completely regular space and Y is a disconnected metrizable separable space, then the following conditions are equivalent:

1. X is almost strongly zero-dimensional;
2. $\mathscr{F}_\sigma^*(X,Y) = \mathrm{B}_1(X,Y)$.

Y is metrizable

- A family $\mathcal{A} = (A_i : i \in I)$ of subsets of a topological space X is called *strongly functionally discrete*, if there exists a discrete family $(U_i : i \in I)$ of cozero subsets of X such that $\overline{A_i} \subseteq U_i$ for every $i \in I$.
- A family $\mathcal{B}$ of sets of a topological space X is called *a base* for a map $f : X \to Y$ if the preimage $f^{-1}(V)$ of an arbitrary open set V in Y is a union of sets from $\mathcal{B}$.
- If $\mathcal{B}$ is a countable union of strongly functionally discrete families, we say that f is σ-strongly functionally discrete, $f \in \Sigma^s(X, Y)$.

If Y is metrizable and separable space, then

every function $f : X \to Y$ is σ-strongly functionally discrete.

Y is metrizable

Theorem (K., 2017)

If X is a completely regular space with $\dim X = 0$ and Y is a metrizable space, then

$$\mathscr{F}_\sigma^*(X,Y) \cap \Sigma^s(X,Y) = \mathrm{B}_1(X,Y).$$

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A question

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Properties of ASZD spaces

Theorem (K., 2022)

① $\dim X = 0 \;\Rightarrow\; X$ is ASZD $\;\Rightarrow\; X$ is totally separated

② If X is countably compact or X is a continuous image of a Polish space, then X is ASZD $\Leftrightarrow \dim X = 0$.

③ If X is a perfectly normal with $\dim X = 0$ and $\varphi : X \to \mathbb{R}$ is piecewise continuous. Then the graph $\Gamma_\varphi \subseteq X \times \mathbb{R}$ is ASZD.

Y is not metrizable

Theorem (W. Rudin, 1981)

If X is a metrizable space, Y is a topological space and Z is a locally convex space, then

$$CB_\alpha(X \times Y, Z) \subseteq B_{\alpha+1}(X \times Y, Z).$$

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Question (O. Sobchuk and V. Mykhaylyuk, 1995)

Is every function $f \in CB_1([0,1] \times [0,1], [0,1])$ a pointwise limit of separately continuous functions $f_n : [0,1] \times [0,1] \to [0,1]$?

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$$\Updownarrow$$

Question (T. Banakh)

$$\mathscr{F}_\sigma([0,1], C_p([0,1])) = B_1([0,1], C_p([0,1])) \ ?$$

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$$\mathscr{F}_\sigma([0,1], C_p([0,1])) \subset B_2([0,1], C_p([0,1]))$$

HOMOTOPIC BAIRE-1 CLASS

An equivalent definition of the fist Baire class

Definition

We say that $f \in \mathrm{B}_1(X, Y)$ if there exists a continuous map $H : X \times \omega \to Y$ such that $f(x) = \lim_{n\to\infty} H(x, n)$ for every $x \in X$.

The first homotopic Baire class

Definition

We say that $f \in \mathrm{hB}_1(X,Y)$ if there exists a continuous map $H : X \times [0,+\infty) \to Y$ such that $f(x) = \lim_{n\to\infty} H(x,n)$ for every $x \in X$.

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If Y is contractible, then $\mathrm{B}_1(X,Y) = \mathrm{hB}_1(X,Y)$.

The first homotopic Baire class

Question (S. Maksymenko).

Let S^1 be the unit circle in $\mathbb{C}$. Is it true that $\mathrm{B}_1(S^1, S^1) = \mathrm{hB}_1(S^1, S^1)$?

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General problem

To describe classes of spaces X and Y such that $\mathrm{B}_1(X, Y) = \mathrm{hB}_1(X, Y)$.

B_1-Lifting property

A continuous map $f : X \to Y$ is a **weak local homeomorphism** if $\forall y \in Y$ $\exists V \ni y, U \subseteq X$ such that $f|_U : U \to V$ is a homeomorphism.

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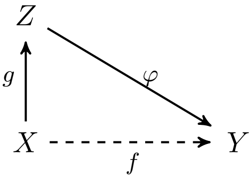
Assume that X, Y and Z are topological spaces and $\varphi : Z \to Y$ is a weak local homeomorphism. We say that the triple (X, Y, Z) has **$\mathscr{P}$-Lifting Property** whenever for all $f \in \mathscr{P}(X, Y)$ there exists $g \in \mathscr{P}(X, Z)$ such that $f = \varphi \circ g$.



Results and questions

Lifting Theorem for B_1-functions (K. and Maksymenko, 2020)

Let X, Y, Z be topological spaces and Y is a paracompact space weakly covered by a metrizable path-connected and locally path-connected space Z. Then (X, Y, Z) has B_1-Lifting Property.

Results and questions

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Theorem (K. and Maksymenko, 2020)

Any open path-connected subset of a normed space is weakly covered by a contractible and locally contractible metrizable space.

Results and questions

Theorem (K. and Maksymenko, 2020)

Let X be a topological space and Y be a path-connected metrizable ANR. Then

$$\mathrm{B}_1(X,Y) = h\mathrm{B}_1(X,Y).$$

Results and questions

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Do there exists a path-connected subset $X \subseteq \mathbb{R}^2$ such that $\mathrm{B}_1(X, X) \neq \mathrm{hB}_1(X, X)$?

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$$f \text{ is a uniform limit of a sequence of } f_n \in \mathrm{hB}_1(X,X)$$

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$$f \text{ is a uniform limit of a sequence of } f_n \in \mathrm{hB}_1(X, X)$$
$$\Downarrow$$

Question 2

Assume that $X \subseteq \mathbb{R}^2$ is a path-connected space. Is it true that $\mathrm{hB}_1(X, X)$ is closed under uniform limits?

FRAGMENTABILITY

Definition

Let X be a topological space, (Y, d) be a metric space and $\varepsilon > 0$.

A function $f : X \to Y$ is *fragmented*, if for every $\varepsilon > 0$ it is ε-*fragmented*, i.e. there exists a sequence $\mathscr{U} = (U_\xi : \xi \in [0, \alpha))$ in X of open sets such that

- $\operatorname{diam} f(U_{\xi+1} \setminus U_\xi) < \varepsilon$ for all $\xi \in [0, \alpha)$;
- $\emptyset = U_0 \subset U_1 \subset U_2 \subset \dots$;
- $U_\gamma = \bigcup_{\xi<\gamma} U_\xi$ for every limit ordinal $\gamma \in [0, \alpha)$.

We call α *an index of* ε-*fragmentability of* f.

Theorem (Jayne, Orihuela, Pallarés and Vera, 1992)

Let X be a perfectly paracompact hereditarily Baire space, Y be a convex subset of a Banach space. The following are equivalent:

- f is fragmented;
- f is of the first Baire class.

Functionally fragmented maps

A function $f : X \to Y$ is *fragmented*, if for every $\varepsilon > 0$ there exists a sequence $\mathscr{U} = (U_\xi : \xi \in [0, \alpha))$ in X of open sets such that

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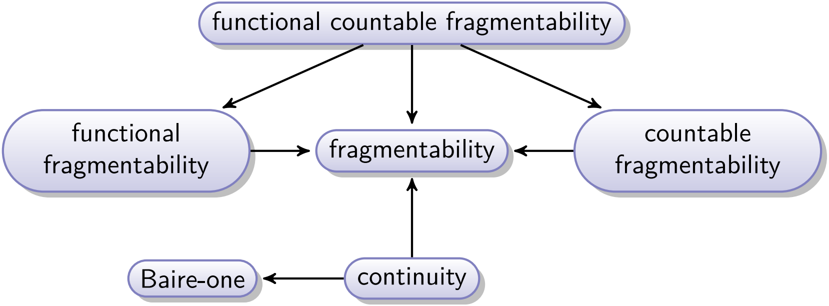
An ε-fragmented map $f : X \to Y$ is

- *functionally ε-fragmented* if every U_ξ is a cozero set in X;
- *functionally ε-countably fragmented* if $\mathscr{U}$ can be chosen to be countable;
- *functionally countably fragmented* if f is functionally ε-countably fragmented for all $\varepsilon > 0$.

Functionally fragmented maps



Relations between different types of fragmentability

1. Let X be a topological space, (Y, d) be a metric space, $\varepsilon > 0$ and $f : X \to Y$ be a map. If one of the following conditions holds
   - Y is separable and f is continuous,
   - X is hereditarily Lindelöf and f is fragmented,
   - X is compact and $f \in \mathrm{B}_1(X, Y)$,
   - X is Lindelöf, $f \in \mathrm{B}_1(X, Y)$ and fragmented,
   - X is Lindelöf, f is functionally fragmented,

   then f is functionally countably fragmented.

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    - X is Lindelöf, $f \in \mathrm{B}_1(X, Y)$ and fragmented,
    - X is Lindelöf, f is functionally fragmented,

   then f is functionally countably fragmented.
2. If one of the following conditions holds
    - f is functionally countably fragmented,
    - X is perfectly paracompact and f is fragmented,
    - X is paracompact and f is functionally fragmented,

   then $f \in B_1(X, \mathbb{R})$.

Relations between different types of fragmentability

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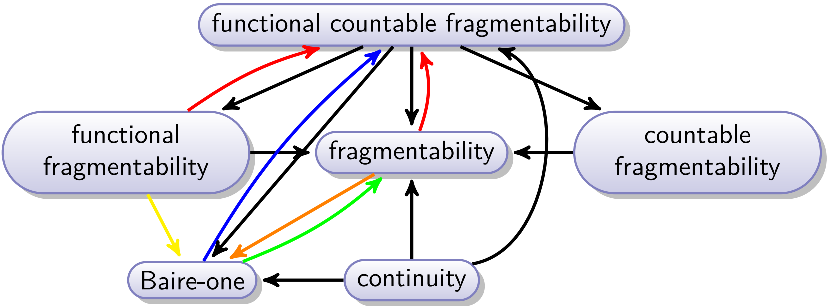
   then $f \in B_1(X, \mathbb{R})$.
3. If X is hereditarily Baire and $f \in B_1(X, \mathbb{R})$, then f is fragmented.

Further relations ($Y = \mathbb{R}$)



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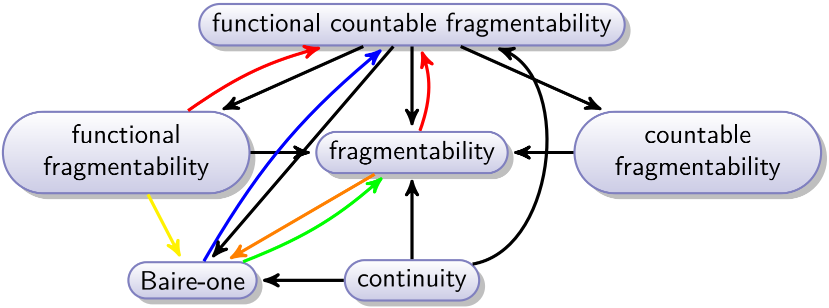

- X is compact
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- X is perfectly paracompact
- X is hereditarily Baire
- X is paracompact

Question

Let X be paracompact, $f : X \to \mathbb{R}$ be fragmented and $f \in B_1$. Is f functionally fragmented?

Application of fragmentability to extension of B_1-functions

Theorem (O. Kalenda and J. Spurný, 2005)

Let E be a Lindelöf subspace of a completely regular space X and $f : E \to \mathbb{R}$ be a Baire-one function. If

- E is G_δ, or
- E is hereditarily Baire,

then there exists a Baire-one function $g : X \to \mathbb{R}$ such that $g = f$ on E.

Questions

1. Let X be a hereditarily Baire completely regular space and f a Baire-one function on X. Can f be extended to a Baire-one function on βX?
2. Let X be a normal space, Y a closed hereditarily Baire subset of X and f a Baire-one function on Y. Can f be extended to a Baire-one function on X?
3. Let X be a normal space, $Y = \bigcap_{n\in\omega} G_n \subseteq X$ is an intersection of co-zero sets and f a Baire-one function on Y. Can f be extended to a Baire-one function on X?

Application of fragmentability to extension of B_1-functions

Theorem (K. and Mykhaylyuk, 2020)

Let X be a completely regular space and $f : X \to \mathbb{R}$ be a Baire-one function. Consider the following conditions:

(i) f is functionally countably fragmented,

(ii) f is extendable to a Baire-one function on βX,

(iii) f is extendable to a Baire-one function on any completely regular space $Y \supseteq X$,

(iv) f is extendable to a Baire-one function on any compactification Y of X,

(v) f is fragmented.

Then (i) $\Leftrightarrow$ (ii).

If X is Lindelöff, then

$$\text{(i)} \Leftrightarrow \text{(ii)} \Leftrightarrow \text{(iii)} \Leftrightarrow \text{(iv)} \Leftrightarrow \text{(v)}$$

Corollary

Question 1 (O. Kalenda and J. Spurny)

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Theorem (K. and Mykhaylyuk, 2020)

There exist a completely metrizable locally compact space X and a Baire one function $f : X \to [0, 1]$ such that f is not countably fragmented, in particular, f can not be extended to a Baire one function $g : \beta X \to [0, 1]$.

Corollary

- For every ordinal $\alpha \in [\omega, \omega_1)$ there exists a function $f : [0,1] \to [0,1]$ such that the index of the 1-fragmentability of f is $\alpha + 1$.

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- For every $\alpha < \omega_1$ we put $X_\alpha = [0,1]$ and consider the completely metrizable locally compact space $X = \bigoplus_{\alpha<\omega_1} X_\alpha$.

  Now for every $\alpha < \omega_1$ we choose a countably fragmented function $f_\alpha : X_\alpha \to [0,1]$ such that the index of fragmentability of f is greater than α. Now we consider the function $f : X \to [0,1]$, $f(x) = f_\alpha(x)$ if $x \in X_\alpha$. Since every f_α is a Baire one function, f is a Baire one function too. Moreover, it is clear that f is not countably fragmented.

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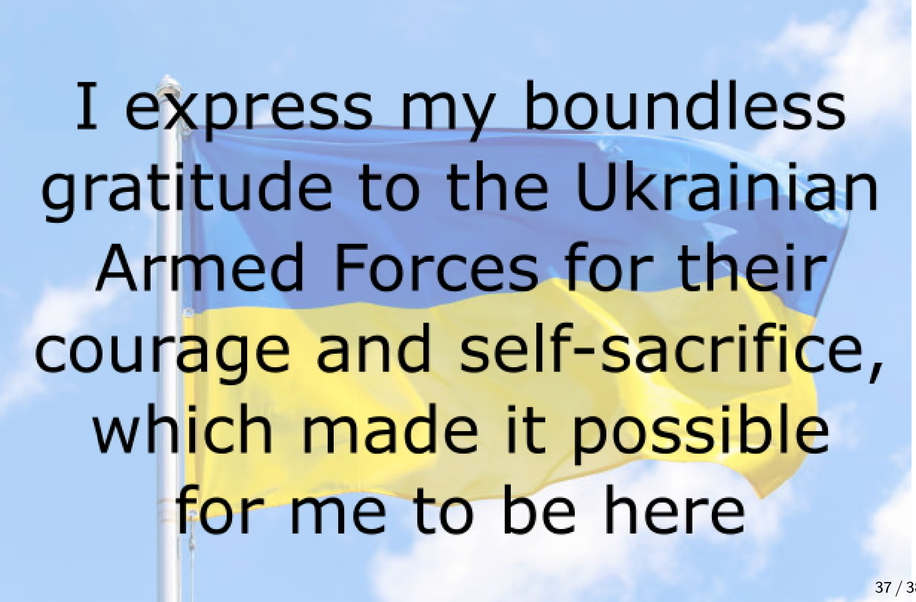

POSTSCRIPTUM

Thank you for the attention!

Glory to Ukraine!