Recognizing the topologies on subspaces in L^p-spaces on metric measure spaces

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Outline

1. Topological types of subspaces in L^p-spaces
2. Z-sets in $L^p(X)$
3. Characterizations of compact sets in L^p-spaces

Topological types of subspaces in L^p-spaces

Throughout this talk, $X = (X, d, \mu)$ is a metric measure space satisfying the following:

- (Borel) Borel sets of X are measurable;
- (Borel-regular) ${}^\forall E \subset X$ is contained in a Borel set $B \subset X$ s.t. $\mu(E) = \mu(B)$;
- For ${}^\forall x \in X$ and ${}^\forall r \in (0, \infty)$, $0 < \mu(B(x, r)) < \infty$.

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For $1 \leq p < \infty$, let $L^p(X) = (L^p(X), \|\cdot\|_p)$ be the L^p-space on X, which is a Banach space.

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Due to the efforts of R.D. Anderson and M.I. Kadec, we have the following:

Theorem 1.1

If X is infinite and separable, then $\mathrm{L}^{\mathrm{p}}(X) \approx \ell_2$.

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Theorem A (K (2020))

If X is separable and locally compact, and $\{x \in X \mid \mu(\{x\}) \neq 0\}$ is not dense in X, then $\mathrm{UC}(X) \approx (\ell_2^f)^{\mathbb{N}}$.

A space X is **doubling** if the following is satisfied.

- $^{\exists}\gamma \geq 1$ s.t. $\mu(B(x, 2r)) \leq \gamma\mu(B(x, r))$ for $^{\forall}x \in X$ and $^{\forall}r > 0$.

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Theorem B (K (?))

Let X be non-degenerate, separable and doubling. Suppose that

$(\star)$ for ${}^{\forall}x \in X$,

$$(0, \infty) \ni r \mapsto \mu(B(x, r)) \in (0, \infty)$$

is continuous.

Then $(\mathrm{L}^\mathrm{p}(X), \mathrm{LIP}_b(X)) \approx (\ell_2 \times \mathbf{Q}, \ell_2^f \times \mathbf{Q})$.

Z-sets in $L^p(X)$

Definition (Z-set)

For $A \underset{closed}{\subset} Y$, A is a **(strong)** Z-**set** in Y if id_Y is approximated by $f : Y \to Y$ s.t. $f(Y) \cap A = \emptyset$ $(\mathrm{cl}\, f(Y) \cap A = \emptyset)$.

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For a class $\mathfrak{C}$, Y is **strongly $\mathfrak{C}$-universal** if the following holds.

- Let $f : A \to Y$, $A \in \mathfrak{C}$. Suppose that $B \underset{closed}{\subset} A$ and $f|_B$ is a Z-embedding. Then f is approximated by a Z-embedding $g : A \to Y$ s.t. $g|_B = f|_B$.

A Z-embedding is an embedding whose image is a Z-set.

Theorem 2.1 (absorbing set)

Let $Y \subset M$. Suppose that $M \approx \ell_2$ and Y satisfies the following:

1. *Y is homotopy dense in M and $Y \in (\mathfrak{M}_2)_\sigma$;*
2. *Y is strongly $\mathfrak{M}_2$-universal;*
3. *Y is contained in a strong Z_σ-set in M.*

Then $Y \approx (\ell_2^f)^{\mathbb{N}}$.

Y is **homotopy dense** in M if $^\exists h : M \times \mathbf{I} \to M$ s.t. $h(M \times (0,1]) \subset Y$ and $h(y,0) = y$ for $^\forall y \in M$.

$\mathfrak{M}_2$ is the class of absolute $F_{\sigma\delta}$ sets.

A strong Z_σ-set is a countable union of strong Z-sets.

Problem

Detect Z-sets in $\mathrm{L}^p(X)$.

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Lemma 2.2

Let $\phi : Y \to \mathrm{L^p}(X)$ *and* $a \in X$ *with* $\mu(\{a\}) = 0$. *Then for* ${}^\forall\epsilon : Y \to (0,1)$, ${}^\exists\psi : Y \to \mathrm{L^p}(X)$ *and* ${}^\exists\delta : Y \to (0,1)$ *s.t. for* ${}^\forall y \in Y$,

1. $\|\phi(y) - \psi(y)\|_p \leq \epsilon(y)$,
2. $\psi(y)(B(a, \delta(y))) = \{0\}$.

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Let $\phi : Y \to \mathrm{L^p}(X)$ and $a \in X$ with $\mu(\{a\}) = 0$. Then for $^\forall \epsilon : Y \to (0,1)$, $^\exists \psi : Y \to \mathrm{L^p}(X)$ and $^\exists \delta : Y \to (0,1)$ s.t. for $^\forall y \in Y$,

1. $\|\phi(y) - \psi(y)\|_p \leq \epsilon(y)$,
2. $\psi(y)(B(a, \delta(y))) = \{0\}$.

For $^\forall n \in \mathbb{N}$ and $^\forall U \underset{open}{\subset} \operatorname{int}\{x \in X \mid \mu(\{x\}) = 0\}$,

$$Z(n, U) = \{f \in \mathrm{L^p}(X) \mid |f(x)| \geq 1/n \text{ for a.e. } x \in U\}$$

is a Z-set in $\mathrm{L^p}(X)$ by Lemma 2.2.

Lemma 2.3

Let $a \in X$ *with* $\mu(\{a\}) = 0$. *Suppose that* $A \subset \mathrm{L}^{\mathrm{p}}(X)$ *and* $\xi : A \to (0, \infty)$ *s.t. for* ${}^{\forall} f \in A$, $f(x) = 0$ *for a.e.* $x \in B(a, \xi(f))$, *and that* B *is a* Z*-set in* $\mathrm{L}^{\mathrm{p}}(X)$. *If* $A \cup B \underset{\text{closed}}{\subset} \mathrm{L}^{\mathrm{p}}(X)$, *then it is a* Z*-set.*

Characterizations of compact sets in L^p-spaces

Theorem 3.1 (D. Curtis-T. Dobrowolski-J. Mogilski (1984))

Let C be a σ-compact convex set in a completely metrizable linear space. Suppose that $\operatorname{cl} C$ is an AR and not locally compact. Then $(\operatorname{cl} C, C) \approx (\ell_2 \times \mathbf{Q}, \ell_2^f \times \mathbf{Q})$ if C contains an infinite-dimensional locally compact convex set.

Compact sets in ℓ_2 are Z-sets.

Problem

Give a criterion for subsets of $L^p(X)$ to be compact.

For $f : \mathbb{R}^n \to \mathbb{R}$ and $a \in \mathbb{R}^n$, $\tau_a f(x) = f(x - a)$.

For $E \subset X$, χ_E is the characteristic function of E, and for $f : X \to \mathbb{R}$, $f\chi_E(x) = f(x) \cdot \chi_E(x)$.

Problem

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Theorem 3.2 (A.N. Kolmogorov (1931), M. Riesz (1933))

A subset $F \underset{\text{bounded}}{\subset} \mathrm{L}^p(\mathbb{R}^n)$ is relatively compact if and only if the following are satisfied.

1. *For $^\forall\epsilon > 0$, $^\exists\delta > 0$ s.t. $\|\tau_a f - f\|_p < \epsilon$ for $^\forall f \in F$ and $^\forall a \in \mathbb{R}^n$ with $|a| < \delta$.*
2. *For $^\forall\epsilon > 0$, $^\exists r > 0$ s.t. $\|f\chi_{\mathbb{R}^n \setminus B(\mathbf{0},r)}\|_p < \epsilon$ for $^\forall f \in F$.*

For $f : \mathbb{R}^n \to \mathbb{R}$ and $a \in \mathbb{R}^n$, $\tau_a f(x) = f(x - a)$.

For $E \subset X$, χ_E is the characteristic function of E, and for $f : X \to \mathbb{R}$, $f\chi_E(x) = f(x) \cdot \chi_E(x)$.

For $f \in \mathsf{L^p}(X)$ and $r > 0$, define $A_r f : X \to \mathbb{R}$ by

$$A_r f(x) = \frac{1}{\mu(B(x, r))} \int_{B(x,r)} f \chi_{B(x,r)}(y) d\mu(y),$$

which is called the **average function** of f.

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which is called the **average function** of f.

Theorem 3.3 (P. Górka-A. Macios (2014))

Let X be doubling and $p > 1$. Suppose that $\inf\{\mu(B(x,r)) \mid x \in X\} > 0$ for $^\forall r > 0$. Then $F \underset{bounded}{\subset} \mathrm{L^p}(X)$ is relatively compact if and only if the following hold.

1. *For $^\forall \epsilon > 0$, $^\exists \delta > 0$ s.t. for $^\forall f \in F$ and $^\forall r \in (0,\delta)$, $\|A_r f - f\|_p < \epsilon$.*
2. *For $^\forall \epsilon > 0$, $^\exists E \underset{bounded}{\subset} X$ s.t. $\|f\chi_{X \setminus E}\|_p < \epsilon$ for $^\forall f \in F$.*

Theorem C

Let X be doubling. Suppose that

$(*)$ for ${}^\forall x \in X$ and ${}^\forall r > 0$,

$$\mu(B(x,r) \triangle B(y,r)) \to 0 \text{ as } y \to x.$$

Then $F \underset{bounded}{\subset} \mathrm{L}^{\mathrm{p}}(X)$ is relatively compact if and only if the following are satisfied.

1. For ${}^\forall \epsilon > 0$, ${}^\exists \delta > 0$ s.t. for ${}^\forall f \in F$ and ${}^\forall r \in (0,\delta)$, $\|A_r f - f\|_p < \epsilon$.
2. For ${}^\forall \epsilon > 0$, ${}^\exists E \underset{bounded}{\subset} X$ s.t. $\|f \chi_{X \setminus E}\|_p < \epsilon$ for ${}^\forall f \in F$.

For $A, B \subset X$, let $A \triangle B = (A \setminus B) \cup (B \setminus A)$.

Consider the following conditions between d and μ.

($\star$) For ${}^\forall x \in X$,

$$(0, \infty) \ni r \mapsto \mu(B(x, r)) \in (0, \infty)$$

is continuous.

($*$) For ${}^\forall x \in X$ and ${}^\forall r \in (0, \infty)$,

$$\mu(B(x, r) \triangle B(y, r)) \to 0 \text{ as } y \to x.$$

($\dagger$) For ${}^\forall r \in (0, \infty)$,

$$X \ni x \mapsto \mu(B(x, r)) \in (0, \infty)$$

is continuous.

Then ($\star$) $\Rightarrow$ ($*$) $\Rightarrow$ ($\dagger$).

Fix $^{\forall}x_0 \in X$. For $n \in \mathbb{N}$, let

$$L(n) = \{f \in \mathrm{LIP}_b(X) \mid \|f\|_p \leq n, \mathrm{lip}\, f \leq n, \mathrm{supp}\, f \subset B(x_0, n)\}.$$

Then $\mathrm{LIP}_b(X) = \bigcup_{n \in \mathbb{N}} L(n)$.

$\mathrm{lip}\, f$ is the lipschitz constant of f and $\mathrm{supp}\, f$ is the support of f.

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Then $\mathrm{LIP}_b(X) = \bigcup_{n\in\mathbb{N}} L(n)$.
If X is doubling and satisfies $(*)$, then $L(n)$ is compact by Theorem C.

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