Arbitrarily Large Countably Compact Free Abelian Groups

Matheus Koveroff Bellini (USP)

Co-authors: Artur Hideyuki Tomita (USP), Klaas Pieter Hart (UDelft), Vinicius de Oliveira Rodrigues (USP)

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Some History

Previous results on $\mathbb{Z}^{(\mathfrak{c})}$ being countably compact:

- Tkachenko, 1990 : Under CH
- Tomita, 1998 : Under MA(σ-centered)
- Koszmider, Tomita, Watson, 2000 : Under MA(countable), forcing example
- Madariaga-García, Tomita, 2007 : Under $\mathfrak{c}$ selective ultrafilters (also $\mathbb{Z}^{(2^{\mathfrak{c}})}$, under $2^{\mathfrak{c}}$ selective ultrafilters)
- Boero, Castro-Pereira, Tomita, 2019 : Under 1 selective ultrafilter

Some History

Two results on finite powers:

- Boero, Tomita, 2011 : The square is countably compact, under $\mathfrak{c}$ selective ultrafilters
- Tomita, 2015 : All finite powers are countably compact, under $\mathfrak{c}$ incomparable selective ultrafilters

We also recall that in [8] Tomita showed that the ω-th power of a free Abelian group cannot be countably compact.

The Setting

We have obtained the following:

Theorem

Assume that there are $\mathfrak{c}$ incomparable selective ultrafilters. Then for every cardinal κ such that $\kappa^\omega = \kappa$, there is a Hausdorff group topology on the free Abelian group of cardinality κ without non-trivial convergent sequences and whose finite powers are countably compact.

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Recall the following result:

Theorem (van Douwen, [3])

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This classifies, under GCH, which free Abelian groups allow a countably compact group topology:

Theorem (GCH)

A free Abelian group of infinite cardinality κ can be endowed with a countably compact group topology if and only if $\kappa = \kappa^\omega$.

Notation

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- Given an ultrafilter p and $f, g \in (\mathbb{Q}^{(\kappa)})^\omega$ we denote $f \equiv_p g$ if $\{n \in \omega : f(n) = g(n)\} \in p$, and the class of f in this equivalence relation $[f]_p$. The quotient $(\mathbb{Q}^{(\kappa)})^\omega / \equiv_p$ is here denoted $\mathrm{Ult}(\mathbb{Q}, p)$ and is called the *ultrapower* of $\mathbb{Q}^{(\kappa)}$ by p. We note that $\mathrm{Ult}(\mathbb{Q}, p)$ has a natural $\mathbb{Q}$-vector space structure.

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- Given $\xi < \kappa$, $\chi_{\vec{\xi}}$ is the sequence constantly equal to $\chi_{\{\xi\}} \in \mathbb{Z}^{(\kappa)}$.
- The unit circle group $\mathbb{T}$ is the metric group $(\mathbb{R}/\mathbb{Z}, \delta)$, with the metric $\delta(x + \mathbb{Z}, y + \mathbb{Z}) = \min\{|x - y + n| : n \in \mathbb{Z}\}$.

Selective Ultrafilters and the Rudin-Keisler Order

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- Given an ultrafilter p and an $f : \omega \to \omega$, let $f_*(p) = \{A \in \omega : f^{-1}[A] \in p\}$.
- We say that $p \leq_{\mathrm{RK}} q$ if there exists an $f : \omega \to \omega$ such that $p = f_*(q)$.
- We say that p and q are *incomparable* if neither $p \leq_{\mathrm{RK}} q$ or $q \leq_{\mathrm{RK}} p$.

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- If $T = (A_i : i < k) \in \mathcal{T}$, then $l(T) = A_{k-1}$ (and $l(\varnothing) = \omega$).

Lemma

Assume there are $\mathfrak{c}$ incomparable selective ultrafilters. Then there is a family of incomparable selective ultrafilters $(p_{T,n} : T \in \mathcal{T}, n \in \omega)$ such that $l(T) \in p_{T,n}$ whenever $T \in \mathcal{T}$ and $n \in \omega$.

Branching Linear Independence

Definition

Let $\mathcal{F}$ be a subset of G^ω and $A \in [\omega]^\omega$. We shall call $\mathcal{F}$ *linearly independent mod* A^* if for every free ultrafilter p with $A \in p$,

$$([f]_p : f \in \mathcal{F}) \cup ([\chi_{\vec{\xi}}]_p : \xi < \kappa)$$

is a linearly independent family of the $\mathbb{Q}$-vector space $\mathrm{Ult}(\mathbb{Q}, p)$.

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Lemma

Every set of sequences that is l.i. mod A^* can be extended to a maximal linearly independent set mod A^*.

Lemma

Let g be an element of G^ω and let $\mathcal{E} \subseteq G^\omega$ be maximal l.i. mod B^*. Then there exist an infinite subset A of B, a finite subset E of $\mathcal{E}$, a finite subset D of κ, and sets $\{r_f : f \in E\}$ and $\{s_\nu : \nu \in D\}$ of rational numbers such that

$$g|_A = \sum_{f \in E} r_f \cdot f|_A + \sum_{\nu \in D} s_\nu \cdot \chi_{\vec{\nu}}|_A.$$

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$$g|_A = \sum_{f \in E} r_f \cdot f|_A + \sum_{\nu \in D} s_\nu \cdot \chi_{\vec{\nu}}|_A.$$

Corollary

If $\mathcal{E} \subseteq G^\omega$ is maximal l.i. mod B^*, then $|\mathcal{E}| = \kappa$.

Branching Linear Independence

Proposition

There exists a family $(\mathcal{E}_T : T \in \mathcal{T})$ *such that:*

1. *For every* $T \in \mathcal{T}$ *the set* $\mathcal{E}_T$ *is maximal l.i. mod* $I(T)^*$*, and*
2. *For every* $T \in \mathcal{T}$*, if* $n \leq |T|$ *then* $\mathcal{E}_{T|n} \subseteq \mathcal{E}_T$*.*

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So now we enumerate each $\mathcal{E}_T$ faithfully as $\{f_\xi^T : \kappa \leq \xi < \kappa + \kappa\}$.

Generators of Free Subgroups

Definition

For each $T \in \mathcal{T}$ and $n \in \omega$, we denote by $G_{T,n}$ the intersection of $\mathrm{Ult}(\mathbb{Z}, p_{T,n})$ and the free Abelian group generated by $\{\frac{1}{n!}[f_\xi^T]_{p_{T,n}} : \kappa \leq \xi < \kappa + \kappa\} \cup \{\frac{1}{n!}[\chi_{\vec{\xi}}]_{p_{T,n}} : \xi < \kappa\}$.

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Lemma

The group $G_{T,n}$ has a basis of the form $\{[\chi_{\vec{\xi}}]_{p_{T,n}} : \xi < \kappa\} \cup \{[f]_{p_{T,n}} : f \in \mathcal{F}_{T,n}\}$ for some subset $\mathcal{F}_{T,n}$ of G^ω.

Lemma

Assume that for every pair (T, n) in $\mathcal{T} \times \omega$ every sequence f in $\mathcal{F}_{T,n}$ has a $p_{T,n}$-limit in G. Then every finite power of G is countably compact.

Obtaining p-limits via homomorphisms

Enumerate $G^\omega = \{h_\xi : \omega \leq \xi < \kappa\}$ so that $\operatorname{supp} h_\xi(n) \subseteq \xi$ for all $n \in \omega$ and $\omega \leq \xi < \kappa$, with $\mathfrak{c}$ repetitions.

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Lemma

There exists a family $(J_{T,n} : T \in \mathcal{T}, n \in \omega)$ of pairwise disjoint subsets of κ such that $\{h_\xi : \xi \in J_{T,n}\} = \mathcal{F}_{T,n}$.

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Lemma

There exists a family $(J_{T,n} : T \in \mathcal{T}, n \in \omega)$ of pairwise disjoint subsets of κ such that $\{h_\xi : \xi \in J_{T,n}\} = \mathcal{F}_{T,n}$.

The following lemma is the main step towards guaranteeing that each $f \in \mathcal{F}_{T,n}$ has a $p_{T,n}$-limit.

Obtaining p-limits via homomorphisms

Lemma (Countable Homomorphism)

Assume we have $d \in G \setminus \{0\}$, $r \in G^\omega$ *injective, and* $D \in [\kappa]^\omega$ *such that*

1. $\omega \cup \operatorname{supp} d \cup \bigcup_{n\in\omega} \operatorname{supp} r(n) \subseteq D$,
2. $D \cap J_{T,n} \neq \emptyset$ *for infinitely many* (T, n)*'s and,*
3. $\operatorname{supp} h_\xi(n) \subseteq D$ *for all* $n \in \omega$ *and* $\xi \in D \setminus \omega$

Then there exists a homomorphism $\phi : \mathbb{Z}^{(D)} \to \mathbb{T}$ *such that:*

1. $\phi(d) \neq 0$
2. $p_{T,n} - \lim(\phi \circ h_\xi) = \phi(\chi_\xi)$, *whenever* $T \in \mathcal{T}$, $n \in \omega$, *and* $\xi \in D \cap J_{T,n}$
3. $\phi \circ r$ *does not converge.*

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Then there exists a homomorphism $\phi : \mathbb{Z}^{(D)} \to \mathbb{T}$ *such that:*

1. $\phi(d) \neq 0$
2. $p_{T,n} - \lim(\phi \circ h_\xi) = \phi(\chi_\xi)$*, whenever* $T \in \mathcal{T}$*,* $n \in \omega$*, and* $\xi \in D \cap J_{T,n}$
3. $\phi \circ r$ *does not converge.*

From this Lemma we obtain, by recursion, the full homomorphism.

Obtaining p-limits via homomorphisms

Lemma

Assume $d \in G \setminus \{0\}$ and $r \in G^\omega$ is injective. Then there exists a homomorphism $\phi : \mathbb{Z}^{(\kappa)} \to \mathbb{T}$ such that

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Lemma

Assume $d \in G\setminus\{0\}$ and $r \in G^\omega$ is injective. Then there exists a homomorphism $\phi : \mathbb{Z}^{(\kappa)} \to \mathbb{T}$ such that

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2. $p_{T,n} - \lim(\phi \circ h_\xi) = \phi(\chi_\xi)$, *whenever* $T \in \mathcal{T}$, $n \in \omega$ *and* $\xi \in J_{T,n}$
3. $\phi \circ r$ *does not converge.*

The main result follows from obtaining such a $\phi_{d,r}$ for each $d \in G\setminus\{0\}$ and $r \in G^\omega$ injective, and considering the initial topology generated by these $\phi_{d,r}$.

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- The other is the following lemma:

Where Do We Use Incomparableness?

Lemma

Let $(\mathcal{F}^k : k \in \omega)$ be a sequence of countable subsets of G^ω and let $(p_k : k \in \omega)$ be a sequence of pairwise incomparable selective ultrafilters such that for each $k \in \omega$ $([f]_{p_k} : f \in \mathcal{F}^k) \cup ([\chi_{\vec{\xi}}]_{p_k} : \xi \in \kappa)$ is linearly independent. Furthermore let for every $f \in \bigcup_k \mathcal{F}^k$ a $\xi_f \in \kappa$ be given. In addition let $d, d' \in G \setminus \{0\}$ with disjoint supports. Finally, let $D \in [\kappa]^\omega$ containing $\omega \cup \operatorname{supp} d \cup \operatorname{supp} d'$ and $\bigcup_n \operatorname{supp} f(n)$ for every $f \in \bigcup_k \mathcal{F}^k$. Then there exists a homomorphism $\phi : \mathbb{Z}^{(D)} \to \mathbb{T}$ such that

1. $\phi(d) \neq 0$, $\phi(d') \neq 0$ *and*
2. $p_k - \lim(\phi \circ f) = \phi(\chi_{\xi_f})$, *whenever $k \in \omega$ and $f \in \mathcal{F}^k$.*

Where Do We Use Incomparableness?

This last lemma requires the following combinatorial principle:

Lemma

Let $(p_k : k \in \omega)$ be a family of pairwise incomparable selective ultrafilters. For each k let $(a_{k,i} : i \in \omega)$ be a strictly increasing sequence in ω such that $\{a_{k,i} : i \in \omega\} \in p_k$ and $i < a_{k,i}$ for all $i \in \omega$. Then there exists $\{I_k : k \in \omega\}$ such that:

- (a) $\{a_{k,i} : i \in I_k\} \in p_k$, *for each* $k \in \omega$.
- (b) $I_j \cap I_j = \emptyset$ *whenever* $i, j \in \omega$ *and* $i \neq j$, *and*
- (c) $\{[i, a_{k,i}] : i \in I_k \text{ and } k \in \omega\}$ *is a pairwise disjoint family.*

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Thank you for your attention!