The Menger property is ℓ-invariant

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A space X is *Menger* (resp., *Hurewicz*) if for every sequence $(\mathcal{U}_n)_{n\in\mathbb{N}}$ of open covers of X, there is a sequence $(\mathcal{V}_n)_{n\in\mathbb{N}}$ such that for every n, $\mathcal{V}_n$ is a finite subfamily of $\mathcal{U}_n$ and the family $\bigcup_{n\in\mathbb{N}}\mathcal{V}_n$ covers X (resp., every point of X is contained in $\bigcup\mathcal{V}_n$ for all but finitely many n's).

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$$\sigma\text{-compact} \Rightarrow \text{Hurewicz} \Rightarrow \text{Menger} \Rightarrow \text{Lindelöf}.$$

Theorem (Arhangel'skii-Pytkeev, 1982)

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Property $(*)$ is a weaker form of first-countability.

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Player II wins if $\emptyset \neq \bigcap_{n \in \omega} V_n \subseteq bX \setminus X$, otherwise Player I wins.

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Corollary

Player I has no winning strategy in $kP(bX, bX \setminus X) \Leftrightarrow$ the space X is Menger.

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TFAE:

1. X is projectively Hurewicz
2. $\forall F \subseteq \beta X \setminus X$, such that F is a countable union of zero-sets in βX, there exists a G_δ subset G of βX with $F \subseteq G \subseteq \beta X \setminus X$.

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The game $zP(\beta X, \beta X \setminus X)$ is played as $kP(\beta X, \beta X \setminus X)$ with additional requirement that the compact sets played by player I are zero-sets in βX.

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Theorem (Bouziad, 1999)

Suppose that Z is Čech-complete. If C is compact and $\phi : C \to \mathcal{K}(Z)$ is l.s.c., then there is a compact $L \subseteq Z$ that meets every value of ϕ, i.e. $C = \phi^{-1}(L)$