Generic Polish metric spaces

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Joint work with Christian Bargetz, Adam Bartoš, and Franz Luggin.

The Banach-Mazur game

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Consider the following infinite game in which two players alternately build a chain $M_0 \subseteq M_1 \subseteq \cdots$ in $\mathcal{M}$.

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Proposition

A generic space for a given class is determined uniquely (but it may not exist at all).

The Urysohn space

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Theorem (Urysohn 1927)

The Urysohn space $\mathbb{U}$ exists, is determined uniquely, up to isometry. Furthermore, $\mathbb{U}$ contains all separable metric spaces and is homogeneous, namely, every isometry between finite subsets of $\mathbb{U}$ extends to a bijective isometry of $\mathbb{U}$.

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The Urysohn space is generic over the class of all finite metric spaces.

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Given a countable linearly ordered set D with the minimal element 0, there exists a generic space over the class of all finite ultrametric spaces with distances in D.

Generic ultrametric space

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Theorem (Kwiatkowska, Malicki, K.)

There exists a generic ultrametric space $\mathbb{A}$, in the sense of the Banach-Mazur game, where the two players build both the spaces and the distances.
The space $\mathbb{A}$ is homogeneous.

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(H0) $\implies$ (H1).

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Given a metric space M, its age is the class $\mathrm{Age}(M)$ consisting of all finite metric spaces isometric to subspaces of M.

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Theorem

An approximately homogeneous Polish space is generic over its age.

A fake homogeneous space

Example

Take the rational Urysohn space $\mathbb{U}_{\mathbb{Q}}$ (the variant of $\mathbb{U}$, allowing rational distances only) and let $V = \mathbb{U}_{\mathbb{Q}} \times \{0, \sqrt{2}\}$ with the ℓ_1-metric, namely,

$$\varrho(\langle x, i\rangle, \langle y, j\rangle) = \varrho(x, y) + |i - j|,$$

for every $x, y \in \mathbb{U}_{\mathbb{Q}}$, $i, j \in \{0, \sqrt{2}\}$.
Let $\mathcal{F}$ be the class of all finite metric spaces isometric to subsets of V. Clearly, V is homogeneous. On the other hand, if $\overline{V}$ denotes the completion of V, then obviously $\operatorname{Age}(\overline{V})$ is the class of all finite metric spaces and $\overline{V}$ is far from being approximately homogeneous.

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Theorem

There exists a Polish space $\mathbb{T}$, generic over $\mathcal{T}^{\mathbb{Q}}$. It is approximately homogeneous but not homogeneous.

Weak amalgamations

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Definition

Let $\mathcal{M}$ be a class of finite metric spaces. We say that $\mathcal{M}$ has the approximate weak amalgamation property if for every $A \in \mathcal{M}$, for every $\varepsilon > 0$ there exists an embedding $e \colon A \to A'$ such that for every embeddings $f \colon A' \to X$, $g \colon A' \to Y$ there are embeddings $f' \colon X \to B$, $g' \colon Y \to B$ satisfying

$$\varrho(f' \circ f \circ e, g' \circ g \circ e) < \varepsilon.$$

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Theorem

Every class of finite metric spaces admitting a generic space has the approximate weak amalgamation property.

The proper statement

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Theorem (cf. Krawczyk, K. 2021)

Let $\mathcal{M}$ be a class of finite metric spaces. Consider the modified game $\mathrm{BM}(\mathcal{M},\mathcal{U})$*, where the second player wins if the resulting space embeds into some space from the class* $\mathcal{U}$*.*
If $|\mathcal{U}| < 2^{\aleph_0}$ *and the second player has a winning strategy in* $\mathrm{BM}(\mathcal{M},\mathcal{U})$*, then* $\mathcal{M}$ *has the approximate weak amalgamation property.*

Graphs as metric spaces

Proposition

Let $\mathcal{M}$ be a hereditary class of finite metric spaces with distances $\{0, 1, 2\}$, that has the weak amalgamation property and the joint embedding property. Then there exists an $\mathcal{M}$-generic space.

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Question

How about more complicated examples?

Banach spaces

Definition

A Banach space $\mathbb{E}$ is generic over over a class $\mathcal{F}$ of finite-dimensional spaces if the second player has a winning strategy in $\mathrm{BM}\,(\mathcal{F}, \mathbb{E})$, playing with linear isometric embeddings.
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Proposition

The separable Hilbert space is generic over the class of all Euclidean spaces. It is homogeneous.

The quest for new amalgamations

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Theorem (Viscardi 2017, still unpublished)

Let $\mathcal{F}$ be the smallest class of all finite-dimensional normed spaces obtained by using the following two operations:

- *Making the standard amalgamation.*
- *Selecting a subspace.*

Then the Gurarii space is generic over $\mathcal{F}$.

References

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