Complexity of distances, reducibility and universality

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ABSTRACT: We introduce and study the notion of Borel reducibility between pseudometrics on standard Borel spaces, which is a generalization of the famous notion of Borel reducibility between equivalence relations.
The central object of our investigations is the Gromov-Hausdorff distance, which turns out to be equally complex as several other distances between metric or Banach spaces, such as the Kadets distance or the Banach-Mazur distance. Next, we consider the notion of an orbit pseudometric and provide a continuous version of the well-known result of Clemens, Gao and Kechris that the relation of isometry of Polish metric spaces is bireducible with a universal orbit equivalence relation.
The present results come from the collaboration with Marek Cúth and Michal Doucha.

The talk is based on the following papers.

- M. Cúth, M. Doucha and O. Kurka, *Complexity of distances between metric and Banach spaces: Theory of generalized analytic equivalence relations*, to appear in J. Math. Logic, arXiv:1804.11164.
- M. Cúth, M. Doucha and O. Kurka, *Complexity of distances between metric and Banach spaces: Reductions of distances between metric and Banach spaces*, Israel J. Math. **248** (2022), 383–439.
- O. Kurka, *Orbit pseudometrics and a universality property of the Gromov-Hausdorff distance*, preprint, arXiv:2204.08375.

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Our task is to find out if two $n \times n$ matrices A_1 and A_2 are similar. We find their Jordan canonical forms and look whether they are the same up to the order of the Jordan blocks.

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In some sense, we have reduced the problem of similarity of square matrices to the problem whether Jordan matrices are the same.

Definition

Let E and F be equivalence relations on Polish spaces X and Y. We say that E is *Borel reducible* to F, and write

$$E \leq_B F,$$

if there exists a Borel mapping $f : X \to Y$ (so-called *reduction*) such that

$$f(x)\, F\, f(x') \;\Leftrightarrow\; x\, E\, x', \quad x, x' \in X.$$

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Remark

$E \leq_B F$ means that E is at most as "complex" as F.

Definition

By the *Urysohn space* we mean the (up to isometry) only complete separable metric space $\mathbb{U}$ with the property that for any finite metric space A and any isometric embedding $f : B \to \mathbb{U}$, where $B \subseteq A$, there exists an isometric embedding $\widetilde{f} : A \to \mathbb{U}$ extending f.

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It is easy to show that the Urysohn space is isometrically universal for complete separable metric spaces. (For most of the talk, this will be the only fact we need to know about $\mathbb{U}$.)

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It is easy to show that the Urysohn space is isometrically universal for complete separable metric spaces. (For most of the talk, this will be the only fact we need to know about $\mathbb{U}$.)
In fact, the following result holds.

Theorem (Katětov)

Let X be a complete separable metric space. Then there is an isometric embedding $i : X \to \mathbb{U}$ such that any surjective isometry on $i(X)$ can be extended to a surjective isometry on $\mathbb{U}$.

In the following definition, we introduce the Polish space of all complete separable metric spaces.

Definition

We define

$$F(\mathbb{U}) = \{F \subseteq \mathbb{U} : F \text{ is closed}\},$$

and we equip $F(\mathbb{U}) \setminus \{\emptyset\}$ with the *Wijsman topology*, defined as the coarsest topology for which the function

$$F \mapsto \delta_{\mathbb{U}}(u, F),$$

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is continuous for each $u \in \mathbb{U}$.

Fact

$F(\mathbb{U}) \setminus \{\emptyset\}$ equipped with the Wijsman topology forms a Polish space.

Remark

Similarly, we can define the Polish space of all separable Banach spaces. We just equip

$$\mathcal{B} = \{F \subseteq C([0,1]) : F \text{ is closed and linear}\}.$$

with the corresponding Wijsman topology.

With these two codings, we can consider equivalence relations between metric/Banach spaces as equivalence relations on a Polish space.

Definition

Let G be a group with identity e and let X be a set. By a *group action* $G \curvearrowright X$ we mean a mapping $(g, x) \in G \times X \mapsto g \cdot x \in X$ satisfying

$$e \cdot x = x$$

and

$$(gh) \cdot x = g \cdot (h \cdot x).$$

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Definition

Let $G \curvearrowright X$ be a Borel action of a Polish group G on a Polish space X. The corresponding *orbit equivalence relation* is defined by

$$x E_G^X y \quad \Leftrightarrow \quad \exists g \in G : g \cdot x = y.$$

Theorem (Miller, 1977)

The equivalence classes of E_G^X are Borel.

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The universal orbit equivalence is also bireducible with:

- the (linear) isometry relation of separable Banach spaces (Melleray),
- the affine homeomorphism of Choquet simplices (Sabok),
- the isomorphism relation of separable C*-algebras (Sabok),
- the homeomorphism relation of compact metric spaces (Zielinski).

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- the homeomorphism relation of compact metric spaces (Zielinski).

Consequently, all these relations have Borel equivalence classes.

Definition

The equivalence relation E_1 on $\mathbb{R}^{\mathbb{N}}$ is defined by

$$x\, E_1\, y \quad \Leftrightarrow \quad \exists N\, \forall n \geq N : x(n) = y(n).$$

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The relation E_1 is not Borel reducible to the isometry relation of complete separable metric spaces (as well as to the other equivalences from the previous slide).

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The relation E_1 is not Borel reducible to the isometry relation of complete separable metric spaces (as well as to the other equivalences from the previous slide).

The following question is an important open problem.

Question

Let E be a Borel equivalence relation into which E_1 is not Borel reducible. Is E Borel reducible to an orbit equivalence relation then?

Definition

The *Gromov-Hausdorff distance* of non-empty metric spaces M and N is defined by

$$\varrho_{GH}(M,N) = \inf_{\substack{X \text{ metric space} \\ i_M : M \hookrightarrow X \\ i_N : N \hookrightarrow X}} \varrho_H^X(i_M(M), i_N(N))$$

(where i_M, i_N are isometric embeddings of M, N into X and ϱ_H^X denotes the Hausdorff distance between subsets of X).

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The *Kadets distance* of Banach spaces X and Y is defined by

$$\varrho_K(X,Y) = \inf_{\substack{Z \text{ Banach space} \\ i_X : X \hookrightarrow Z \\ i_Y : Y \hookrightarrow Z}} \varrho_H^Z(i_X(B_X), i_Y(B_Y))$$

(where i_X, i_Y are linear isometric embeddings of X, Y into Z).

Definition

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Later, we will give a negative answer to the following question.

Question (Ben Yaacov, Doucha, Nies, Tsankov)

Is $\varrho_{GH}(M, \cdot)$ a Borel function for every $M \in F(\mathbb{U}) \setminus \emptyset$?
Is $\varrho_K(X, \cdot)$ a Borel function for every $X \in \mathcal{B}$?

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It was pointed out by C. Rosendal that any orbit equivalence is reducible to both $E_{\varrho_{GH}}$ and E_{ϱ_K}. The following remains open.

Question (Ben Yaacov, Doucha, Nies, Tsankov)

Is the relation $E_{\varrho_{GH}}$, resp. E_{ϱ_K}, Borel reducible to an orbit equivalence relation?

Theorem 1 (Cúth, Doucha, K.)

The following equivalences are mutually Borel bireducible:

- $E_{\varrho_{GH}}$
- $E_{\varrho_{GH}}$ *restricted to metric spaces with distances in* $\{0\} \cup [1, 2]$
- E_{ϱ_K}
- $E_{\varrho_{BM}}$, *where* ϱ_{BM} *is the Banach-Mazur distance of Banach spaces*
- E_{ϱ_L}, *where* ϱ_L *is the Lipschitz distance of metric spaces*
- $E_{\varrho_L^{\mathcal{B}}}$, *where* $\varrho_L^{\mathcal{B}}$ *is the Lipschitz distance of Banach spaces*
- $E_{\varrho_{HL}}$, *where* ϱ_{HL} *is the Hausdorff-Lipschitz distance of metric spaces*
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- E_{ϱ_N}, *where* ϱ_{HL} *is the net distance of metric spaces*

Corollary

The equivalence classes of the relations $E_{\varrho_{BM}}, E_{\varrho_L}, E_{\varrho_L^{\mathcal{B}}}, E_{\varrho_{HL}}$ *and* E_{ϱ_N} *are Borel.*

Definition (Cúth, Doucha, K.)

Let ϱ_X and ϱ_Y be pseudometrics on Polish spaces X and Y. We say that ϱ_X is *Borel-u.c. reducible* to ϱ_Y, and write

$$\varrho_X \leq_{B,u} \varrho_Y,$$

if there exists a Borel mapping $f : X \to Y$ such that, for every $\varepsilon > 0$, there are $\delta_X > 0$ and $\delta_Y > 0$ satisfying

$$\forall x, y \in X : \quad \varrho_X(x, y) < \delta_X \Rightarrow \varrho_Y(f(x), f(y)) < \varepsilon$$

and

$$\forall x, y \in X : \quad \varrho_Y(f(x), f(y)) < \delta_Y \Rightarrow \varrho_X(x, y) < \varepsilon.$$

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We say that ϱ_X is *Borel-u.c. bireducible* with ϱ_Y if $\varrho_X \leq_{B,u} \varrho_Y$ and $\varrho_Y \leq_{B,u} \varrho_X$.

Remark

The reducibility between pseudometrics is

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- a strengthening of the reducibility between equivalences in the sense that
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- a generalization of the reducibility between equivalences in the sense that
$$\varrho_E \leq_{B,u} \varrho_F \quad \Leftrightarrow \quad E \leq_B F.$$

Here, for an equivalence relation E on a Polish space X, the pseudometric ϱ_E is defined by

$$\varrho_E = \begin{cases} 0, & x\,E\,y, \\ 1, & \text{otherwise.} \end{cases}$$

The proof of Theorem 1 provides also a quantitative version:

Theorem 2 (Cúth, Doucha, K.)

The following distances are mutually Borel-u.c. bireducible:

- ϱ_{GH}
- ϱ_{GH} *restricted to metric spaces with distances in* $\{0\} \cup [1, 2]$
- ϱ_K
- ϱ_{BM}
- ϱ_L
- $\varrho_L^{\mathcal{B}}$
- ϱ_{HL}
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Let $\mathcal{M}_1^2$ denote the subspace of $F(\mathbb{U})$ consisting of all spaces with distances of distinct point in $[1,2]$.

For $M = \{p_n : n \in \mathbb{N}\} \in \mathcal{M}_1^2$, let us consider the following renorming of ℓ_2:

$$\|x\|_M = \sup\left(\{\|x\|_{\ell_2}\} \cup \left\{\frac{1}{\sqrt{2}} \cdot \left(1 + \tfrac{1}{400} \cdot d_M(p_n, p_m)\right) \cdot |x_n + x_m| : n \neq m\right\}\right)$$

for $x = (x_n)_{n=1}^{\infty} \in \ell_2$.

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for $x = (x_n)_{n=1}^\infty \in \ell_2$.

Then the mapping $M \mapsto (\ell_2, \|\cdot\|_M)$, once realized as a Borel mapping $\mathcal{M}_1^2 \to \mathcal{B}$, is a Borel-u.c. reduction of $\varrho_{GH}|_{\mathcal{M}_1^2 \times \mathcal{M}_1^2}$ to the distances ϱ_K, ϱ_{BM}, and the versions of $\varrho_L, \varrho_U, \varrho_{HL}, \varrho_N, \varrho_{GH}$ for $\mathcal{B}$.

Theorem 3 (Cúth, Doucha, K.)

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Corollary

If ϱ is any of the distances $\varrho_{GH}, \varrho_K, \varrho_{BM}, \varrho_L, \varrho_L^{\mathcal{B}}, \varrho_{HL}, \varrho_N$, then the function $\varrho(x,\cdot)$ is not Borel for some x.

A natural class of pseudometrics is obtained by a generalization of orbit equivalence relations:

Definition

Let $G \curvearrowright X$ be a Borel action of a Polish group G on a Polish space X, and let d be a pseudometric on X with the property

$$d(x, y) = d(g \cdot x, g \cdot y), \quad x, y \in X, \ g \in G.$$

Then we define

$$\varrho_{G,d}(x, y) = \inf\{d(g \cdot x, y) : g \in G\}$$

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Remark

In fact, every pseudometric is an orbit pseudometric if we consider a trivial action. For this reason, one should impose some restrictions.

Example

If d is the discrete metric, that is,

$$d(x,y) = \begin{cases} 0, & x = y, \\ 1, & x \neq y, \end{cases}$$

then

$$\varrho_{G,d}(x,y) = \begin{cases} 0, & \exists g \in G : g \cdot x = y, \\ 1, & \text{otherwise.} \end{cases}$$

So, $E_{\varrho_{G,d}}$ is nothing else but the orbit equivalence E_G^X.

Example

Let $X = [1,2]^{[\mathbb{N}]^2}$, let S_∞ denote the group of permutations of $\mathbb{N}$, and let

$$(\pi \cdot x)(m,n) = x\big(\pi^{-1}(m), \pi^{-1}(n)\big)$$

and

$$d_2(x,y) = \sup_{m \neq n} \big| x(m,n) - y(m,n) \big|$$

for $\pi \in S_\infty, x, y \in X, m, n \in \mathbb{N}, m \neq n$.

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It is not difficult to show that $\varrho_{S_\infty, d_2}(x,y) = 2\varrho_{GH}((\mathbb{N},x),(\mathbb{N},y))$ whenever one of the sides is less than 1. It follows:

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It is not difficult to show that $\varrho_{S_\infty, d_2}(x,y) = 2\varrho_{GH}((\mathbb{N}, x), (\mathbb{N}, y))$ whenever one of the sides is less than 1. It follows:

Theorem (Cúth, Doucha, K.)

ϱ_{S_∞, d_2} is Borel-u.c. bireducible with ϱ_{GH}.

Using a method of G. Hjorth, we prove the following generalization of the result of Kechris and Louveau.

Theorem 4 (Cúth, Doucha, K.)

Let G, X and d be as above, and let the action $G \curvearrowright X$ be continuous. If d is a complete metric and generates a topology that is finer than the original topology of X, then E_1 is not Borel reducible to $E_{\varrho_{G,d}}$.

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Considering the result from the previous slide, we obtain:

Corollary

E_1 is not Borel reducible to $E_{\varrho_{GH}}$, E_{ϱ_K}, $E_{\varrho_{BM}}$, E_{ϱ_L}, $E_{\varrho_{HL}}$ and E_{ϱ_N}.

Theorem 5 (K.)

Let G be a Polish group acting continuously on a Polish space X. Let d be a lower semicontinuous pseudometric on X such that $d(x, y) = d(gx, gy)$ for any $x, y \in X$ and $g \in G$.

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Corollary

Let $\varrho_{G,d}$ be as above. Then $E_{\varrho_{G,d}}$ is Borel reducible to E_{GH}, and so its equivalence classes are Borel and E_1 is not Borel reducible to it.

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Corollary

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Theorem 6 (K.)

The Gromov-Hausdorff distance ϱ_{GH} is Borel-u.c. bireducible with the orbit pseudometric $\varrho_{Iso(\mathbb{U}),\varrho_H}$ on $F(\mathbb{U}) \setminus \{\emptyset\}$.

We now present some basic tools and ideas of the proof of Theorem 5.

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We need the following result.

Theorem (Melleray)

Let X be a complete separable metric space of diameter at most 1 and let G be a closed subgroup of $\mathrm{Iso}(X)$. Then there exists an extension Y of X such that

- *Y is a complete separable metric space,*
- *any member of G can be extended in a unique way to a surjective isometry on Y,*
- *any surjective isometry on Y is an extension of a member of G.*

Without loss of generality, $d \leq 1$.
We can also assume that the sequence $s_1, s_2, \dots$ is non-decreasing.
Let γ be a compatible right-invariant metric on G with $\gamma \leq 1$.
Let δ_X be a compatible complete metric on X with $\delta_X \leq 1$.
Let us consider the maximum distance on $G \times X$.

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Then Melleray's theorem provides:

Claim

There is an extension Z of $G \times X$ such that

- *Z is a complete separable metric space of diameter at most 2,*
- *for any $h \in G$, the isometry $I_h : (g, x) \mapsto (gh, x)$ can be extended in a unique way from $G \times X$ to a surjective isometry on Z,*
- *any surjective isometry on Z is an extension of I_h for some $h \in G$.*

We define

$$W = (Z \times \mathbb{N}) \cup (G \times X \times [0,1] \times \mathbb{N})$$

and a metric m on W as follows:

$$m((z_1, k_1), (z_2, k_2)) = 100 \cdot |2^{k_1} - 2^{k_2}| + 2^{\min\{k_1, k_2\}} \delta_Z(z_1, z_2),$$

$$m((z, l), (g, x, u, k)) = u + 10 \cdot 2^k + m((z, l), ((g, x), k)),$$

$$m((g_1, x_1, u_1, k), (g_2, x_2, u_2, k)) = |u_1 - u_2| + 2^k \delta_Z((g_1, x_1), (g_2, x_2)),$$

and for $k_1 \neq k_2$,

$$\begin{aligned} m((g_1, x_1, u_1, k_1), (g_2, x_2, u_2, k_2)) = &u_1 + 10 \cdot 2^{k_1} + u_2 + 10 \cdot 2^{k_2} \\ &+ m(((g_1, x_1), k_1), ((g_2, x_2), k_2)). \end{aligned}$$

For every $p \in X$, let us consider the subspace of W

$$W_p = (Z \times \mathbb{N}) \cup \{(g, x, s_k(gp, x), k) : g \in G, x \in X, k \in \mathbb{N}\}.$$

Let Y be the completion of W and, for every $p \in X$, let Y_p be the closure of W_p in Y.

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The mapping $p \mapsto Y_p$ from X to $F(Y)$ is Borel.

Claim

For all $p, q \in X$,

$$\varrho_{Iso(Y),\varrho_H}(Y_p, Y_q) \leq \varrho_{G,d}(p, q) \leq 2\varrho_{GH}(Y_p, Y_q).$$

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